Section 8.2: Integration by Parts

When you finish your homework, you should be able to…

- Use the integration by parts technique to find indefinite integral and evaluate definite integrals
- Use the tabular method to organize an integral requiring integration by parts
- Recognize trends and establish guidelines for integrals requiring integration by parts

Warm-up:

1. Differentiate with respect to the independent variable.

a. $\frac{d}{dx} f(x) = \frac{d}{dx} \arcsin 5x$

$$f'(x) = \frac{\frac{d}{dx}(5x)}{\sqrt{1-(5x)^2}}$$

$$\boxed{f'(x) = \frac{5}{\sqrt{1-25x^2}}}$$

b. $\frac{d}{dx} y = \frac{d}{dx} \ln(5x+1)$

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(5x+1)}{5x+1}$$

$$\boxed{\frac{dy}{dx} = \frac{5}{5x+1}}$$

c. $\frac{d}{d\theta} r(\theta) = \frac{d}{d\theta} \tan\theta$

$$\boxed{r'(\theta) = \sec^2\theta}$$

2. Find the indefinite integral.

a. $\int \frac{\arctan x}{x^2+1}dx$

$= \int (\arctan x)^1 \cdot \frac{1}{1+x^2}\,dx$

$= \boxed{\frac{\arctan^2 x}{2} + C}$

$g(x) = \arctan x$

$g'(x) = \frac{1}{1+x^2}$

b. $\int \frac{(\ln x)^3}{x}dx$

$= \int (\ln x)^3 \cdot \frac{1}{x}\,dx$

$= \boxed{\frac{(\ln x)^4}{4} + C}$

$g(x) = \ln x$

$g'(x) = \frac{1}{x}$

$\int f[g(x)]\underline{g'(x)dx} = F[g(x)] + C$

c. $\int x\sqrt{5-x}\,dx$

$u = 5 - x \rightarrow x = 5 - u$

$\frac{du}{dx} = -1$

$dx = -du$

$= -\int x\,(u)^{1/2}(-du)$

$= -\int (5-u)u^{1/2}\,du$

$= -\int (5u^{1/2} - u^{3/2})\,du$

$= -\left(\frac{10}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right) + C$

$= \underbrace{-\frac{10}{3}(5-x)^{3/2}} + \frac{2}{5}(5-x)^{5/2} + C$

$= \frac{1}{15}(5-x)^{3/2}\left[-50 + \overset{30-6x}{6(5-x)^1}\right] + C$

$= \frac{1}{15}(5-x)^{3/2}(-20-6x) + C$

$= \frac{-2}{15}(5-x)^{3/2}(10+3x) + C$

3. Evaluate the definite integral.

$$\frac{1}{6}\int_0^{\pi/8} \tan^2 6\theta \, \underline{\sec^2 6\theta d\theta} \cdot 6$$

$$= \frac{1}{6}\int_0^{-1} (u)^2 \, du$$

$$= \frac{1}{6} \cdot \frac{u^3}{3} \Big|_{u=0}^{u=-1}$$

$$= \frac{1}{18}\left((-1)^3 - (0)^3\right)$$

$$= \boxed{-\frac{1}{18}}$$

$u = \tan 6\theta$

$\frac{du}{d\theta} = 6\sec^2 6\theta$

$du = 6\sec^2 6\theta \, d\theta$

$u(\theta) = \tan 6\theta$

$u(\pi/8) = \tan[6 \cdot \pi/8]$

$= \tan 3\pi/4$

$= -1$

$u(0) = \tan[6 \cdot 0]$

$= \tan 0$

$= 0$

__Integration__ by __parts__ is based on the formula for the __derivative__ of a __product__ and is useful for __integrals__ involving products of algebraic and __transcendental__ functions.

Consider the following product of two functions of x that have continuous

derivatives.

$u = f(x)$, $v = g(x)$

$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

$\int d(uv) = \int(u\,dv + v\,du)$

$uv = \int u\,dv + \int v\,du$

$\rightarrow \int u\,dv = uv - \int v\,du$

THEOREM: INTEGRATION BY PARTS

If u and v are functions of x and have continuous derivatives, then

$$\int \underline{u}\,\underline{dv} = uv - \int v\,du$$

This technique turns a super complicated integral into Simpler ones. The trick is to choose your function u so that $\frac{du}{dx}$ is simpler than u. Oh yeah...and **PRACTICE A** BUNCH **OF PROBLEMS!!!**

Okay...let's look at the "easy" types of Integration by Parts (IBP) problems.

Which types of expressions do not have basic integration formulas?

1. logarithmic
2. inverse trig

transcundental function

$$\int u\,dv = uv - \int v\,du$$

EXAMPLE 1: Find the following indefinite integrals.

a. $\int 4x^2 \underbrace{\ln x}\,dx = (\ln x)\left(\frac{4}{3}x^3\right) - \int\left(\frac{4}{3}x^3\right)\left(\frac{dx}{x}\right)$

$= \frac{4}{3}x^3 \ln x - \frac{4}{3}\int x^2\,dx$

$= \frac{4}{3}x^3 \ln x - \frac{4}{3}\cdot\frac{x^3}{3} + C$

$= \frac{4}{9}x^3\left[(3\ln x) - 1\right] + C$

$= \boxed{\frac{4}{9}x^3\left(\ln x^3 - 1\right) + C}$

$u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$du = \frac{dx}{x}$

$\int dv = \int 4x^2\,dx$

$v = \frac{4}{3}x^3$

$$\int u\,dv = uv - \int v\,du$$

b. $\int \arcsin x\,dx = (\arcsin x)x - \int x\,\frac{dx}{\sqrt{1-x^2}}$

$= x\arcsin x + \frac{1}{2}\int x(1-x^2)^{-1/2}\,dx\,(-2)$

$= x\arcsin x + \frac{1}{2}\cdot\frac{(1-x^2)^{1/2}}{\frac{1}{2}} + C$

$= \boxed{x\arcsin x + \sqrt{1-x^2} + C}$

Evil plan

$u = \arcsin x$

$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$

crap!

basic sub. doesn't work

Evil plan #2

IBP

$u = \arcsin x$

$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$

$du = \frac{dx}{\sqrt{1-x^2}}$

$\int dv = \int dx$

$v = x$

$g(x) = 1 - x^2$

$g'(x) = -2x$

When you don't have a transcendental factor, you need to play around with the integrand. Oftentimes it works out to let u be the factor whose derivative is a simpler function than u. Then dv would be the more complicated remaining factor. Use pencil!!! There is a lot of trial and error--especially at first☺

Remember: dv ALWAYS includes dx!

$$\int u\,dv = uv - \int v\,du$$

EXAMPLE 2: Find the indefinite integral.

Evil Plan

IBP

$g(x) = -7x$

$g'(x) = -7$

$u = 6x$

$\frac{du}{dx} = 6$

$du = 6dx$

$\int dv = \int e^{-7x}dx$

$v = -\frac{1}{7}e^{-7x}$

a. $\int \frac{6x}{e^{7x}}dx = \int 6xe^{-7x}dx$

$= (6x)\left(-\frac{1}{7}e^{-7x}\right) - \int\left(-\frac{1}{7}e^{-7x}\right)(6dx)$

$= -\frac{6}{7}xe^{-7x} + \frac{6}{7}\cdot\frac{-1}{7}\int e^{-7x}dx\,(-7)$

$= -\frac{6}{7}xe^{-7x} - \frac{6}{49}e^{-7x} + C$

$= \boxed{-\frac{6}{49e^{7x}}(7x+1) + C}$

$$\int u\,dv = uv - \int v\,du$$

b. $\int x\sqrt{5-x}\,dx = \int x(5-x)^{1/2}\,dx$

$$= x\left(-\tfrac{2}{3}(5-x)^{3/2}\right) - \int\left(-\tfrac{2}{3}(5-x)^{3/2}\right)dx$$

$$= -\frac{2x}{3}(5-x)^{3/2} - \frac{2}{3}\int -(5-x)^{3/2}\,dx$$

$$= -\frac{2x}{3}(5-x)^{3/2} - \frac{2}{3}\cdot\frac{2}{5}(5-x)^{5/2} + C$$

$$= \frac{-2}{15}(5-x)^{3/2}\left[5x + \overset{10-2x}{2(5-x)^{1}}\right] + C$$

$$= \boxed{-\frac{2}{15}(5-x)^{3/2}(3x+10) + C}$$

Evil Plan

IBP

$u = x$ | $\int dv = -\int(5-x)^{1/2}\,dx$ (-1

$du = dx$ | $v = -\frac{2}{3}(5-x)^{3/2}$

$$\int u\,dv = uv - \int v\,du$$

c. $\int x\sec^2 x\,dx = x\tan x - \int \tan x\,dx$

$$\int x\sec^2 x\,dx = x\tan x - (-\ln|\cos x|) + C$$

$$\int x\sec^2 x\,dx = \boxed{x\tan x + \ln|\cos x| + C}$$

Evil Plan

IBP

$u = x$ | $\int dv = \int \sec^2 x\,dx$

$du = dx$ | $v = \tan x$

Sometimes, you need to use IBP multiple times. You may even need to combine like integrals (yes, you can do that)! $\int u\,dv = uv - \int v\,du$

EXAMPLE 3: Find the indefinite integral.

a. $1\int e^{-x}\cos 3x\,dx = (\cos 3x)(-e^{-x}) - \int(-e^{-x})(-3\sin 3x\,dx)$

$\int e^{-x}\cos 3x\,dx = -e^{-x}\cos 3x - 3\int e^{-x}\sin 3x\,dx$

$\int e^{-x}\cos 3x\,dx = -e^{-x}\cos 3x - 3\left[(\sin 3x)(-e^{-x}) - 3\int(-e^{-x})(3\cos 3x\,dx)\right]$

$1\int e^{-x}\cos 3x\,dx = -e^{-x}\cos 3x + 3e^{-x}\sin 3x + 9\int e^{-x}\cos 3x\,dx$

$-8\int e^{-x}\cos 3x\,dx = -e^{-x}\cos 3x + 3e^{-x}\sin 3x + C$

$\int e^{-x}\cos 3x\,dx = -\frac{1}{8}(-e^{-x}\cos 3x + 3e^{-x}\sin 3x) + C$

$\int e^{-x}\cos 3x\,dx = \boxed{\frac{1}{8e^{x}}(\cos 3x - 3\sin 3x) + C}$

Evil Plan

IBP twice
then combine
like integrals

$u_1 = \cos 3x$

$du_1 = -3\sin 3x\,dx$

$\int dv_1 = \int e^{-x}\,dx\,(-1)$

$v_1 = -e^{-x}$

$u_2 = \sin 3x$

$du_2 = 3\cos 3x$

$\int dv_2 = \int e^{-x}\,dx$

$v_2 = -e^{-x}$

b. $\int x^2e^{-x}dx = (x^2)(-e^{-x}) - \int(-e^{-x})(2x\,dx)$

$= -x^2e^{-x} + 2\underbrace{\int xe^{-x}dx}$

$= -x^2e^{-x} + 2\left[(x)(-e^{-x}) - \underbrace{\int -e^{-x}dx}\right]$

$= -x^2e^{-x} - 2xe^{-x} - 2\underbrace{(e^{-x})} + C$

$= \boxed{-\frac{1}{e^x}(x^2+2x+2) + C}$

Evil Plan

IBP 2 times

$u_1 = x^2$ $du_1 = 2x\,dx$

$\int dv_1 = \int e^{-x}dx,\ v_1 = -e^{-x}$

$u_2 = x,\ du_2 = dx$

$dv_2 = e^{-x}dx,\ v_2 = -e^{-x}$

$\int x^2e^{-x}dx$

THE TANZALIN (AKA TABULAR) METHOD is a way of organizing an integration by parts problem. Let's rework the last example using this method.

u

u'

u''

DERIVATIVES u	**INTEGRALS** $dv = e^{-x}dx$ v	**ALTERNATE SIGNS**	**SAME-COLOR PRODUCTS**
x^2			
$2x$	$-e^{-x}$	$+$	$-x^2e^{-x}$
2	e^{-x}	$-$	$-2xe^{-x}$
	$-e^{-x}$	$+$	$-2e^{-x}$

$\int x^2e^{-x}dx = -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C$

$= \boxed{-\frac{1}{e^x}(x^2+2x+2) + C}$

Let's try to bring this all together…

In general, use the following choice for u, in order.

1. Let u equal to any logarithmic or inverse trig factor
2. $u = x^n$
3. $u = e^{ax}$, a is a constant

b is a constant

**When you have $\int e^{ax}\cos bxdx$ or $\int e^{ax}\sin bxdx$, let $dv = e^{ax}dx$ and let $u = \cos bx$ or let $u = \sin bx$.

**To evaluate a definite integral, first find the indefinite integral and then back substitute.

EXAMPLE 5: Find the indefinite integral or evaluate the definite integral.

a. $\int_0^1 x\arcsin x^2 dx$

Evil Plan

IBP

$u = \arcsin x^2$

$du = \frac{2x}{\sqrt{1-x^4}}dx$

$\int dv = \int x dx$

$v = \frac{1}{2}x^2$

Evil plan cont...

$g(x) = 1-x^4$

$g'(x) = -4x^3$

$\int f[g(x)]g'(x)dx = F[g(x)]+C$

Consider: $\int x\arcsin x^2 dx$

$$= \frac{1}{2}x^2\arcsin x^2 - \int\left(\overset{1}{\cancel{\tfrac{1}{2}}}x^2\right)\left(\frac{\overset{1}{\cancel{2}}x\,dx}{\sqrt{1-x^4}}\right)$$

$$= \frac{1}{2}x^2\arcsin x^2 + \frac{1}{4}\int x^3(1-x^4)^{-1/2}dx\,(-4)$$

$$= \frac{1}{2}x^2\arcsin x^2 + \frac{1}{4}\frac{(1-x^4)^{1/2}}{1/2} + C$$

$$= \frac{1}{2}x^2\arcsin x^2 + \frac{1}{2}\sqrt{1-x^4} + C$$

$$\int_0^1 x\arcsin x^2 dx = \left(\frac{1}{2}x^2\arcsin x^2 + \frac{1}{2}\sqrt{1-x^4}\right)\Bigg|_{x=0}^{x=1}$$

$$= \frac{1}{2}\left[\left(1\arcsin 1 + \sqrt{1-1}\right) - \left(\underbrace{\tfrac{1}{2}\cdot 0\arcsin 0}_{0} + \sqrt{1-0}\right)\right]$$

$$= \frac{1}{2}\left(\frac{\pi}{2} - 1\right)$$

$$= \frac{1}{2}\left(\frac{\pi-2}{2}\right) \rightarrow = \boxed{\frac{1}{4}(\pi-2)}$$

b. $\int \frac{x^3 e^{x^2}}{(x^2+1)^2}dx = \int \frac{x \cdot x^2 e^{x^2}}{(x^2+1)^2}dx$

$= x^2 e^{x^2}\left(-\frac{1}{2(x^2+1)}\right) - \int \frac{-1}{\underset{1}{\cancel{2}}\underset{1}{\cancel{(x^2+1)}}}\left(\overset{1}{\cancel{2}}xe^{x^2}\overset{1}{\cancel{(x^2+1)}}dx\right)$

$= -\frac{x^2 e^{x^2}}{2(x^2+1)} + \frac{1}{2}\int x e^{x^2}dx \; (2)$

$= -\frac{x^2 e^{x^2}}{2(x^2+1)} + \frac{1}{2}e^{x^2} \cdot \frac{(x^2+1)}{(x^2+1)} + C$

$= \frac{-x^2 e^{x^2} + x^2 e^{x^2} + e^{x^2}}{2(x^2+1)} + C$

$= \boxed{\frac{e^{x^2}}{2(x^2+1)} + C}$

Evil Plan

use the trick!

IBP

$u = x^2 e^{x^2}$

$du = (2xe^{x^2} + x^2 \cdot 2xe^{x^2})dx$

$du = 2xe^{x^2}(1+x^2)dx$

$du = 2xe^{x^2}(x^2+1)dx$

$\int dv = \frac{1}{2}\int x(x^2+1)^{-2}dx \; (2)$

$g(x) = x^2+1$

$g'(x) = 2x$

$v = \frac{1}{2}\frac{(x^2+1)^{-1}}{-1}$

$v = \frac{-1}{2(x^2+1)}$

Section 8.3: Trigonometric Integrals

When you finish your homework, you should be able to…

π Find indefinite integrals and evaluate definite integrals involving the sine and cosine functions which are raised to positive powers

π Find indefinite integrals and evaluate definite integrals involving the secant and tangent functions which are raised to positive powers

π Use trigonometric identities to find indefinite integral and evaluate definite integrals involving the sine and cosine functions

Warm-up 1: Simplify.

a. $1-\sin^2 x = \cos^2 x$

b. $1+\tan^2 x = \sec^2 x$

c. $\frac{1-\cos 2x}{2} = \sin^2 x$

d. $\frac{1+\cos 2x}{2} = \cos^2 x$

Warm-up 2: Complete the statement.

a. If $u=\sin 2x$, then $du=$ $2\cos 2x\,dx$.

b. If $u=\cos 4x$, then $du=$ $-4\sin 4x\,dx$.

c. If $u=\tan x$, then $du=$ $\sec^2 x$.

d. If $u=\sec 6x$, then $du=$ $6\sec 6x \tan 6x$.

EXAMPLE 1: Find the indefinite integral.

a. $\int \frac{\cos x}{\sqrt{\sin x}}dx = \int \cos x (\sin x)^{-1/2}\,dx$

$$= \frac{(\sin x)^{1/2}}{1/2} + C$$

$$= \boxed{2\sqrt{\sin x} + C}$$

$g(x) = \sin x$

$g'(x) = \cos x$

b. $\int \sin^3 x \cos^2 x dx$

$= \int \sin^2 x \cos^2 x \sin x\, dx$

$= \int (1-\cos^2 x)\cos^2 x \sin x\, dx$

$\overset{-}{=} \int (\cos x)^2 \sin x\, dx(-1) + \int (\cos x)^4 \sin x\, dx(-1)$

$= \boxed{-\frac{(\cos x)^3}{3} + \frac{(\cos x)^5}{5} + C}$

$\sin^3 x(1-\sin^2 x)$

$\sin^3 x - \sin^5 x$

great idea! we don't have $\frac{d}{dx}\sin x$ to use

$g(x) = \cos x$

$g'(x) = -\sin x dx$

So we discovered that if the sine portion of the integrand has an odd, positive integer as a power and the cosine portion has any other power, then we save one sine factor, and convert the others to cosine factors. Then expand and integrate.

$g(x) = \sin 2x$

$g'(x) = 2\cos 2x$

c. $\int \sin^4 2x \cos^3 2x dx = \int (\sin 2x)^4 (\cos 2x)^2 \cos 2x dx$

$$= \frac{1}{2}\int (\sin 2x)^4 (1 - \sin^2 2x) \cos 2x dx \ (2)$$

$$= \frac{1}{2}\left[\int (\sin 2x)^4 2\cos 2x dx - \int (\sin 2x)^6 2\cos 2x dx\right]$$

$$= \frac{1}{2}\left[\frac{(\sin 2x)^5}{5} - \frac{(\sin 2x)^7}{7}\right] + C$$

$$= \boxed{\frac{1}{10}\sin^5 2x - \frac{1}{14}\sin^7 2x + C}$$

Notes

$\int \cos x dx = \sin x + C$

$\cos^2 x = 1 - \sin^2 x$

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

$\int \sin x dx = -\cos x + C$

$\frac{d}{dx}\sin x = \cos x$

etc...

So we discovered that if the cosine portion of the integrand has an odd, positive integer as a power and the sine portion has any other power, then we save one cosine factor, and convert the others to sine factors. Then expand and integrate.

$(A-B) = A^2 - 2AB + B^2$

d. $\int \sin^4 5x dx$

$= \frac{1}{4}\int (1 - 2\cos 10x)^2 dx$

$= \frac{1}{4}\int (1 - 4\cos 10x + \cos^2 10x) dx$

$= \frac{1}{4}\left[\int 1 dx - \frac{4}{10}\int \cos 10x dx^{(10)} + \frac{1}{2}\int (1 + \cos 20x) dx\right]$

$= \frac{1}{4}\left[\int 1 dx - \frac{2}{5}\int \cos 10x \, 10 dx + \frac{1}{2}\int 1 dx + \frac{1}{20 \cdot 2}\int \cos 20x dx \cdot 20\right]$

$= \frac{1}{4}\left[x - \frac{2}{5}\sin 10x + \frac{1}{2}x + \frac{1}{40}\sin 20x\right] + C$

$= \frac{1}{4}x - \frac{1}{10}\sin 10x + \frac{1}{8}x + \frac{1}{160}\sin 20x + C$

$= \boxed{\frac{3}{8}x - \frac{1}{10}\sin 10x + \frac{1}{160}\sin 20x + C}$

~~Evil plan~~

~~$g(x) = \sin 5x$~~

~~$g'(x) = 5\cos 5x$~~

CRAP!!!

Evil Plan 2

- Power-reducing identity

$[\sin^2 5x]^2 = \left[\frac{1 - \cos 2 \cdot 5x}{2}\right]^2$

since $\sin^4 5x = (\sin^2 5x)^2$

$\cos^2 10x = \frac{1 + \cos 2 \cdot 10x}{2}$

$g_1(x) = 10x$

$g_1(x) = 10$

$g_2(x) = 20x$

$g_2'(x) = 20$

So we discovered that if only one sine or cosine factor is in the integrand and has an even, positive integer as a power, you use the power reducing formula

$\sin^2 A = \frac{1 - \cos 2A}{2}$ or $\cos^2 A = \frac{1 + \cos 2A}{2}$ until you can use basic integration formulas. What should we do if both the sine and cosine are raised to even, positive powers?

$\int \sin^2 x \cos^2 x dx$

$\int \sin^2 x (1 - \sin^2 x) dx$

$\int (\sin^2 x - \sin^4 x) dx$

$= \int \left[\frac{1}{2}(1 - \sin 2x) - (\sin^2 x)^2\right] dx$

$= \frac{1}{2}\int (1 - \cancel{\sin 2x} \overset{1-\cos 2x}{} dx - \int \left(\frac{1 - \cos 2x}{2}\right)^2 dx$

etc...

e. $\int \sec^4 x \tan^3 x dx$

$= \int (\sec^2 x)^2 \tan^3 x \, dx$

$= \int \underline{\sec^2 x} \tan^3 x \underline{\sec^2 x \, dx}$

$= \int \underline{(1+\tan^2 x)} \tan^3 x \sec^2 x \, dx$

$= \int (\tan x)^3 \sec^2 x \, dx + \int (\tan x)^5 \sec^2 x \, dx$

$= \frac{(\tan x)^4}{4} + \frac{(\tan x)^6}{6} + C$

$= \boxed{\frac{\tan^4 x}{4} + \frac{\tan^6 x}{6} + C}$

Note:

$\int \sec^2 x \, dx = \tan x + C$

$\int \tan x \, dx = -\ln|\cos x| + C$

$\ast$ $\tan x = \frac{\sin x}{\cos x}$ so $g(x) = \cos x$

$1 + \tan^2 x = \sec^2 x$

$\sec^2 x = 1 + \tan^2 x$

$\tan^2 x = \sec^2 x - 1$

If $g(x) = \tan x$
$g'(x) = \sec^2 x$

If $g(x) = \sec x$
$g'(x) = \sec x \tan x$

Evil plan...

$\sec^4 x = (\sec^2 x)^2$

$g(x) = \tan x$

$g'(x) = \sec^2 x$

So we discovered that if the __secant__ portion of the integrand has an even, positive integer as a power and the tangent portion has any other exponent, then we save __2__ __secant__ factors, and convert the rest to __tangent__ factors. Then __expand__ and __integrate__.

f. $\int \sec^3 5x \tan^3 5x dx$

Evil Plan

If $g(x) = \sec 5x$

$g'(x) = 5\sec 5x \tan 5x$

$$= \int \sec^2 5x \underline{\tan^2 5x} \sec 5x \tan 5x dx (5) \cdot \frac{1}{5}$$

$$= \frac{1}{5} \int \sec^2 5x \underline{(\sec^2 5x - 1)} \cdot 5\sec 5x \tan 5x dx$$

$$= \frac{1}{5} \int (\sec^4 5x - \sec^2 5x) \cdot 5\sec 5x \tan 5x dx$$

$$= \frac{1}{5}\left[\int (\sec 5x)^4 \cdot 5\sec 5x \tan 5x dx - \int (\sec 5x)^2 \cdot 5\sec 5x \tan 5x dx \right]$$

$$= \frac{1}{5}\left[\frac{(\sec 5x)^5}{5} - \frac{(\sec 5x)^3}{3} \right] + C$$

$$= \boxed{\frac{1}{25}\sec^5 5x - \frac{1}{15}\sec^3 5x + C}$$

So we discovered that if the __Secant__ portion of the integrand has an odd, positive integer as a power and the ~~secant~~ tangent portion has any other odd positive exponent, then we save a __secant__-__tangent__ factor, and convert the rest to __secant__ factors. Then __expand__ and __integrate__.

g. $\int \tan^4 x dx$

$= \int (\tan^2 x)^2 dx$

$= \int (\tan^2 x)(\tan^2 x) dx$ ← using to get our g'

$= \int \tan^2 x (\sec^2 x - 1) dx$

$= \int (\tan x)^2 \sec^2 x dx - \int \tan^2 x dx$

$= \int (\tan x)^2 \sec^2 x dx - \int (\sec^2 x - 1) dx$

$= \int (\tan x)^2 \sec^2 x dx - \int \sec^2 x dx + \int 1 dx$

$= \frac{(\tan x)^3}{3} - \tan x + x + C$

$$= \boxed{\frac{1}{3}\tan^3 x - \tan x + x + C}$$

Evil Plan

$$\boxed{\int \sec^2 x dx = \tan x + C}$$

$\tan^2 x = \sec^2 x - 1$

1st integral $\begin{cases} g(x) = \tan x \\ g'(x) = \sec^2 x \end{cases}$

So we discovered that if there is only a tangent factor raised to a positive, even power, rewrite as two tangent factors, one of which is squared, convert the ~~other~~ tangent squared factor to secant squared minus 1, and then expand and integrate.

set up for $\int \tan^8 x dx = \int \tan^6 x \tan^2 x dx$

$= \int (\tan x)^6 (\sec^2 x - 1) dx \rightarrow = \int (\tan x)^6 \sec^2 x dx - \int (\tan x)^6 dx$

1 time ~~twice~~

Evil plan

IBP after splitting secant factors

$u_1 = \sec x$, $du_1 = \sec x \tan x dx$

$\int dv_1 = \int \sec^2 x dx$, $v_1 = \tan x$

h. $\int \sec^3 x dx = \int (\sec x) \sec^2 x dx$

$\int \sec^3 x dx = \sec x \tan x - \int \tan x \sec x \tan x dx$

$\int \sec^3 x dx = \sec x \tan x - \int \sec x \tan^2 x dx$

$\int \sec^3 x dx = \sec x \tan x - \int \sec x (\sec^2 x - 1) dx$

$1 \int \sec^3 x dx = \sec x \tan x - 1 \int \cancel{\sec^3 x dx} + \int \sec x dx$

$+ 1 \int \sec^3 x dx \qquad\qquad + 1 \cancel{\int \sec^3 x dx}$

combining like integrals

$2 \int \sec^3 x dx = \sec x \tan x + \int \sec x dx$

$\frac{1}{2} 2 \int \sec^3 x dx = \frac{1}{2} [\sec x \tan x + \ln|\sec x + \tan x| + C_1]$

$\int \sec^3 x dx = \frac{1}{2} (\sec x \tan x + \ln|\sec x + \tan x|) + C$

$$\boxed{\int \sec^3 x dx = \frac{1}{2} \sec x \tan x + \ln \sqrt{|\sec x + \tan x|} + C}$$

So we discovered that if there is only a secant factor raised to a positive, odd power, we need to use integration by parts, and combine like integrals. If the odd power is >3, you'll be using IBP more than 1 time.

If none of these techniques work, try converting all factors to sine and cosine factors. Then play around with identities.

EXAMPLE 2: Find the indefinite integral.

a. $\int \frac{\tan^2 x}{\sec^5 x}dx = \int \tan^2 x \sec^{-5} x\, dx$

$$= \int \frac{\sin^2 x}{\cos^2 x}\cdot\left(\frac{1}{\cos x}\right)^{-5} dx$$

$$= \int \frac{\sin^2 x}{\cos^2 x}\cdot \frac{1}{\frac{1}{\cos^5 x}}\, dx$$

$$= \int \frac{\sin^2 x}{\cancel{\cos^2} x}\cdot \frac{\cos^{\overset{3}{\cancel{5}}} x}{1}\, dx$$

$$= \int \sin^2 x \cdot \cos^2 x \cos x\, dx$$

$$= \int \sin^2 x \left(1-\sin^2 x\right)\cos x\, dx$$

$$= \int \sin^2 x \cos x\, dx - \int \sin^4 x \cos x\, dx$$

$$= \frac{(\sin x)^3}{3} - \frac{(\sin x)^5}{5} + C$$

$$= \boxed{\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C}$$

Evil Plan

1) If $g(x) = \sec x$
$g'(x) = \sec x \tan x$
crap

2) Convert
$\tan^2 x = \sec^2 x - 1$
but when I expand I'll have negative integer powers of $\sec x$
CRAP!

3). Convert every factor to sines and cosines

- Reserve 1 factor of $\cos x$ for $g'(x)$
- Convert $\cos^2 x$ to $1-\sin^2 x$
- expand and integrate

$g(x) = \sin x$
$g'(x) = \cos x$

$$\int f[g(x)]g'(x)\,dx = F[g(x)] + C$$

$$\int u^n\,du = \frac{u^{n+1}}{n+1},\ n \neq -1$$

b. $\int \frac{\sin^2 x - \cos^2 x}{\cos x} dx = \int \frac{\sin^2 x}{\cos x} dx - \int \frac{\cos^2 x}{\cos x} dx$

$= \int \frac{1-\cos^2 x}{\cos x} dx - \int \cos x \, dx$

$= \int \frac{1}{\cos x} dx \underline{- \int \cos x \, dx - \int \cos x \, dx}$

$= \int \sec x \, dx - 2\int \cos x \, dx$

$= \boxed{\ln|\sec x + \tan x| - 2\sin x + C}$

Evil Plan(s)

1) Is $\sin^2 x - \cos^2 x = 1$

NO! CRAP

2) Split into 2 fractions since $\frac{\cos^2 x}{\cos x} = \cos x$ and $\sin^2 x = 1 - \cos^2 x$ and then split that fraction

PRODUCT TO SUM IDENTITIES

If 2 different angles occur in the integral, use the following identities.

$$\sin mx \sin nx = \frac{1}{2}\left(\cos[(m-n)x] - \cos[(m+n)x]\right)$$

$$\sin mx \cos nx = \frac{1}{2}\left(\sin[(m-n)x] + \sin[(m+n)]x\right)$$

$$\cos mx \cos nx = \frac{1}{2}\left(\cos[(m-n)x] + \cos[(m-n)]x\right)$$

You do not need to memorize these identities.

EXAMPLE 3: Find the indefinite integral.

$$\int \sin 7x \cos 4x dx = \frac{1}{2}\int\left(\sin[(7-4)x] + \sin[(7+4)]x\right)dx$$

$$= \frac{1}{2}\left[\frac{1}{3}\int \sin 3x dx \cdot 3 + \frac{1}{11}\int \sin 11x dx \cdot 11\right]$$

$$= \frac{1}{2}\left[\frac{1}{3}\left(-\cos(3x)\right) + \frac{1}{11}\left(-\cos(11x)\right)\right] + C$$

$$= \boxed{-\frac{1}{6}\cos 3x - \frac{1}{22}\cos 11x + C}$$

Evil Plan

product to sum identity

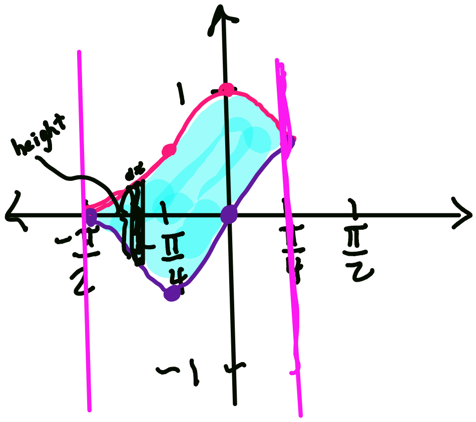
$g_1(x) = 3x$

$g_1'(x) = 3$

$g_2(x) = 11x$

$g_2'(x) = 11$

EXAMPLE 4: Find the area of the region bounded by the graphs of $y = \cos^2 x$, $y = \sin x \cos x$, $x = -\frac{\pi}{2}$, and $x = \frac{\pi}{4}$.



$$A = \int_{-\frac{\pi}{2}}^{\pi/4} (\cos^2 x - \sin x \cos x)\, dx$$

$g_1(x) = \sin x$
$g_1'(x) = \cos x$
1

$$A = \int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \cos^2 x\, dx - \int_{-\pi/2}^{\pi/4} (\sin x) \cos x\, dx$$

$g_2(x) = 2x$
$g_2'(x) = 2$

$$A = \frac{1}{2}\int_{-\pi/2}^{\pi/4} (1 + \underbrace{\cos 2x}_{2})\, dx \cdot 2 - \frac{(\sin x)^2}{2} \Bigg|_{x=-\frac{\pi}{2}}^{x=\pi/4}$$

$$A = \frac{1}{2}\left(x + \frac{\sin(2x)}{2}\right)\Bigg|_{x=-\pi/2}^{x=\pi/4} - \frac{1}{2}\left[(\sin \pi/4)^2 - (\sin(-\pi/2))^2\right]$$

$$A = \frac{1}{2}\left[\left(\frac{\pi}{4} + \frac{\sin(2\cdot\pi/4)}{2}\right) - \left(-\frac{\pi}{2} + \overbrace{\frac{\sin(2(-\pi/2))}{2}}^{0}\right)\right] - \frac{1}{2}\left(\frac{1}{2} - 1\right)$$

$$A = \frac{1}{2}\left(\frac{3\pi}{4} + \frac{1}{2} - 0\right) - \frac{1}{2}\left(-\frac{1}{2}\right)$$

$$A = \frac{3\pi}{8} + \frac{1}{4} + \frac{1}{4}$$

$$A = \frac{3\pi}{8} + \frac{4}{8}$$

$$\boxed{A = \frac{1}{8}(3\pi + 4) \text{ sq. units}}$$

Section 8.4: Trigonometric Substitution

When you finish your homework, you should be able to…

- Find indefinite integrals using trigonmometric substitution
- Evaluate definite integrals using trigonmometric substitution

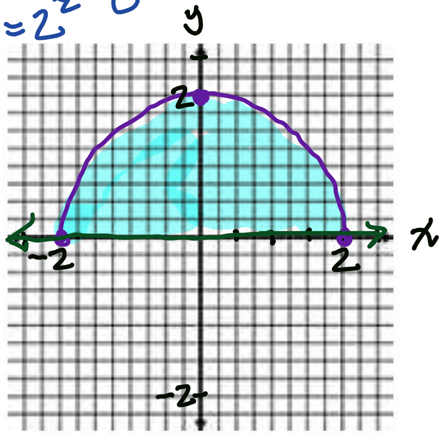
Warm-up 1: Consider the definite integral $\int_{-2}^{2}\sqrt{4-x^2}\,dx$. Do you have the skills to evaluate this definite integral? Yes!

What tool did we use in Calculus I? geometry!

$y = \sqrt{4-x^2} \to y^2 = 4-x^2 \to x^2+y^2 = 2^2$

$\text{Area} = \frac{\pi r^2}{2} = \frac{\pi \cdot 2^2}{2} = 2\pi$

$\int_{-2}^{2}\sqrt{4-x^2}\,dx = 2\pi$

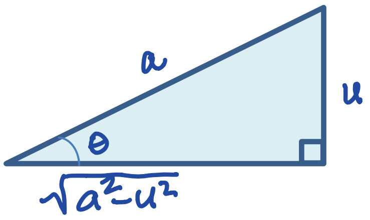


Warm-up 2: Complete the figures.

Note: $(a\sin\theta)^2 = a^2\sin^2\theta$

a. $u = a\sin\theta \to \sin\theta = \frac{u}{a}$

$du = a\cos\theta\, d\theta$



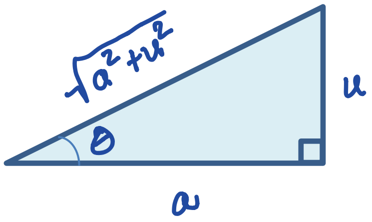
So, $\sqrt{a^2-u^2} = \sqrt{a^2-(a\sin\theta)^2}$

$= \sqrt{a^2(1-\sin^2\theta)}$

$= a\sqrt{\cos^2\theta} \to = a\cos\theta$

for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

b. $u = a\tan\theta \rightarrow \tan\theta = \frac{u}{a} \rightarrow \theta = \arctan\frac{u}{a}$



So, $\sqrt{a^2+u^2} = \sqrt{a^2+(a\tan\theta)^2}$

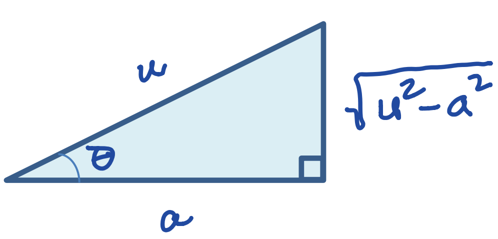
$$= \sqrt{a^2+a^2\tan^2\theta}$$

$$= \sqrt{a^2(1+\tan^2\theta)}$$

$$= \sqrt{a^2\sec^2\theta} \rightarrow = a\sec\theta$$

for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

c. $u = a\sec\theta \rightarrow \sec\theta = \frac{u}{a}$



So, $\sqrt{u^2-a^2} = \sqrt{(a\sec\theta)^2-a^2}$

$$= \sqrt{a^2\sec^2\theta - a^2}$$

$$= \sqrt{a^2(\sec^2\theta-1)}$$

$$= \sqrt{a^2\tan^2\theta}$$

$$= a\tan\theta$$

$$\sqrt{u^2-a^2} = \begin{cases} \underline{\quad a\tan\theta \quad} & \text{for } u > a, \text{ where } 0 \le \theta < \frac{\pi}{2} \\ \underline{\quad -a\tan\theta \quad} & \text{for } u < -a, \text{ where } \frac{\pi}{2} < \theta \le \pi. \end{cases}$$

NOTE: These are the same intervals over which the arcsine, arctangent, and arcsecant are defined. The restrictions on θ ensure that the function used for the substitution is one-to-one.

EXAMPLE 1: Evaluate the definite integral.

$$\int_{-2}^{2}\sqrt{4-x^2}\,dx$$

$u_1 = x,\ du_1 = dx$

$a^2 = 4 \to a = 2$

$dx = du$

$x = u = 2\sin\theta \to \sin\theta = \frac{u}{2}$

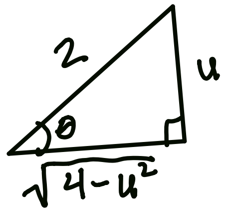
$du = 2\cos\theta\, d\theta$

$\theta = \arcsin\frac{u}{2}$

$u = x$

$du = dx$

$\cos\theta = \frac{\sqrt{4-u^2}}{2}$



$\sqrt{a^2-u^2} = \sqrt{2^2-(2\sin\theta)^2}$

$= \sqrt{4-4\sin^2\theta}$

$= \sqrt{4(1-\sin^2\theta)}$

$= 2\sqrt{\cos^2\theta}$

$= 2\cos\theta$

$g(x) = 2\theta$

$g'(x) = 2$

$\int \sqrt{4-x^2}\,dx = \int 2\cos\theta\, 2\cos\theta\, d\theta$

$= 4\int \cos^2\theta\, d\theta$

$= 4\int \frac{1+\cos 2\theta}{2}\, d\theta$

$= 2\int \left(1+\frac{2\cos 2\theta}{2}\right) d\theta$

$= 2\left[\int 1\, d\theta + \frac{1}{2}\int 2\cos 2\theta\, d\theta\right]$

$= 2\left[\theta + \frac{1}{2}\sin(2\theta)\right] + C$

$= 2\left[\arcsin\frac{u}{2} + \frac{1}{\cancel{2}}\cdot \cancel{2}\sin\theta\cos\theta\right] + C$

$= 2\left[\arcsin\frac{u}{2} + \frac{u}{2}\cdot\frac{\sqrt{4-u^2}}{2}\right] + C$

$= 2\left[\arcsin\frac{x}{2} + \frac{x\sqrt{4-x^2}}{4}\right] + C$

$$\int_{-2}^{2}\sqrt{4-x^2}\,dx = 2\left[\arcsin\frac{x}{2}+\frac{x\sqrt{4-x^2}}{4}\right|_{x=-2}^{x=2}$$

$$= 2\left[\left(\arcsin\frac{2}{2}+\overbrace{\frac{2\sqrt{4-2^2}}{4}}^{0}\right)-\left(\arcsin\frac{-2}{2}+\overbrace{\frac{-2\sqrt{4-(-2)}}{4}}^{0}\right)\right]$$

$$= 2\left(\arcsin 1-\arcsin(-1)\right)$$

$$= 2\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right)$$

$$= 2(\pi)$$

$$= \boxed{2\pi}$$

So we discovered that if the integrand has a $\sqrt{a^2-u^2}$ or $\sqrt{a^2-[f(x)]^2}$ and no basic integration rules, IBP, or regular trigonometric integrals work, we use the substitution $u = a\sin\theta$ or $f(x) = a\sin\theta$.

EXAMPLE 2: Find the indefinite integral.

a. $\int \frac{\sqrt{x^2-16}}{x}dx = \int \frac{\sqrt{(x)^2-(4)^2}}{x}dx$

$= \int \frac{4\tan\theta}{\underset{1}{\cancel{4}}\underset{1}{\cancel{\sec\theta}}}(\overset{1}{\cancel{4}}\overset{1}{\cancel{\sec\theta}}\tan\theta\, d\theta)$

$= 4\int \tan^2\theta\, d\theta$

$= 4\int (\sec^2\theta - 1)\, d\theta$

$= 4\left[\int \sec^2\theta\, d\theta - \int 1\, d\theta\right]$

$= 4[\tan\theta - \theta] + C$

$= 4\left[\frac{\sqrt{x^2-16}}{4} - \operatorname{arcsec}\frac{x}{4}\right] + C$

$= \boxed{\sqrt{x^2-16} - 4\operatorname{arcsec}\frac{x}{4} + C}$

Evil Plan

CRAP!

1) Basic sub doesn't work

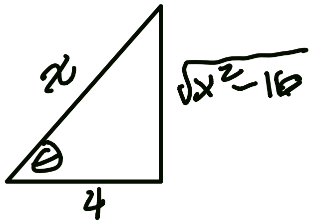
2) IBP doesn't seem to make it easier

3) Trig sub. $\theta = \operatorname{arcsec}\frac{x}{4}$

$a = 4$

$f(x) = x = 4\sec\theta, \sec\theta = \frac{x}{4}$

$dx = 4\sec\theta\tan\theta\, d\theta$



$\sqrt{(x)^2-(4)^2} = \sqrt{(4\sec\theta)^2-(4)^2}$

$= \sqrt{16\sec^2\theta - 16}$

$= \sqrt{16(\sec^2\theta - 1)}$

$= \sqrt{16\tan^2\theta}$

$= 4\tan\theta$

4) Trig. Integral

$\tan^2\theta = \sec^2\theta - 1$

So we discovered that if the integrand has a $\sqrt{u^2-a^2}$ or $\sqrt{[f(x)]^2-a^2}$ and no basic integration rules, IBP, or regular trigonometric integrals work, we use the substitution $u = a\sec\theta$ or $f(x) = a\sec\theta$.

b. $\int \frac{1}{x\sqrt{9x^2+1}}dx = \int \frac{dx}{x\sqrt{1^2+(3x)^2}}$

$= \int \frac{\cancel{\frac{1}{3}}\sec^{\cancel{2}\,1}\theta\, d\theta}{(\cancel{\frac{1}{3}}\tan\theta)\cancel{\sec\theta}}$

$= \int \frac{\sec\theta}{\tan\theta}\, d\theta$

$= \int \frac{1}{\cancel{\cos\theta}}\cdot\frac{\cancel{\cos\theta}}{\sin\theta}\, d\theta$

$= \int \csc\theta\, d\theta$

$= -\ln\left|\csc\theta + \cot\theta\right| + C$

$= -\ln\left|\frac{\sqrt{1+9x^2}}{3x} + \frac{1}{3x}\right| + C$

$= \boxed{-\ln\left|\frac{1+\sqrt{1+9x^2}}{3x}\right| + C}$

$= \ln\left|\left(\frac{1+\sqrt{1+9x^2}}{3x}\right)^{-1}\right| + C$

$= \boxed{\ln\left|\frac{3x}{1+\sqrt{1+9x^2}}\right| + C}$

Evil Plan

~~$u=3x$~~
~~$du=3dx$~~
too many subs.

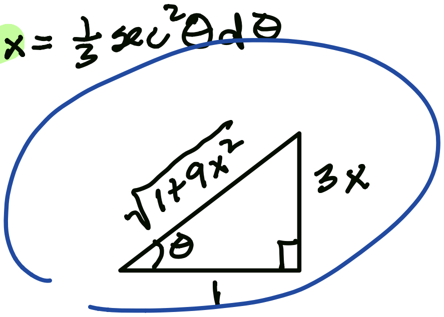
1) Trig Sub.

$a^2=1 \rightarrow a=1$

$f(x)=3x = 1\tan\theta, \tan\theta=\frac{3x}{1}$

$x=\frac{1}{3}\tan\theta$

$dx=\frac{1}{3}\sec^2\theta\, d\theta$



$\sqrt{1+(3x)^2} = \sqrt{1+(1\tan\theta)^2}$

$= \sqrt{1+1\tan^2\theta}$

$= \sqrt{\sec^2\theta}$

$= \sec\theta$

2) Used reciprocal and quotient identities to write integrand in terms of sine and cosine

So we discovered that if the integrand has a $\sqrt{a^2+u^2}$ or $\sqrt{a^2+[f(x)]^2}$ and no basic integration rules, IBP, or regular trigonometric integrals work, we use the substitution $u=a\tan\theta$ or $f(x)=a\tan\theta$

c. $\int \frac{x^2}{\sqrt{2x-x^2}}dx = \int \frac{x^2\, dx}{\sqrt{(1)^2-(x-1)^2}}$

$= \int \frac{(1+\sin\theta)^{2}(\cancel{\cos\theta}\, d\theta)}{\underset{1}{\cancel{\cos\theta}}}$

$= \int (1+\sin\theta)^2\, d\theta$

$= \int (1+2\sin\theta+\sin^2\theta)\, d\theta$

$= \int 1\, d\theta + 2\int \sin\theta\, d\theta + \frac{1}{2}\int (1-\cos 2\theta)\, d\theta$

$= \theta - 2\cos\theta + \frac{1}{2}\int 1\, d\theta - \frac{1}{2}\int \underset{2}{\underline{\cos 2\theta\, d\theta}}^{\,2}$

$= \theta - 2\cos\theta + \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C$

$= \frac{3}{2}\theta - 2\cos\theta - \frac{1}{\underset{2}{\cancel{4}}}\overset{1}{\cancel{2}}\sin\theta\cos\theta + C$

$= \frac{3}{2}\arcsin(x-1) - 2\frac{\sqrt{1-(x-1)^2}}{1} - \frac{1}{2}\cdot\frac{x-1}{1}\cdot\frac{\sqrt{1-(x-1)^2}}{1} + C$

$= \boxed{\frac{3}{2}\arcsin(x-1) - 2\sqrt{1-(x-1)^2} - \frac{1}{2}(x-1)\sqrt{1-(x-1)^2} + C}$

$= \frac{3}{2}\arcsin(x-1) - \frac{\sqrt{1-(x-1)^2}}{2}(4+(x-1)) + C$

$= \boxed{\frac{1}{2}\left(3\arcsin(x-1) - \sqrt{1-(x-1)^2}(3+x)\right) + C}$

Evil Plan

1) complete the square to setup trig. sub.

$-(x^2-2x+(-1)^2)+1$

$= 1-(x-1)^2$

$= (1)^2-(x-1)^2$

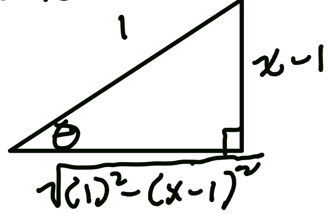
2) trig sub

$a=1,\ f(x)=x-1$

$x-1 = 1\sin\theta,\ \sin\theta = \frac{x-1}{1}$

$x = 1+\sin\theta \qquad \theta = \arcsin(x-1)$

$dx = \cos\theta\, d\theta$



$\sqrt{(1)^2-(x-1)^2} = \sqrt{1-(1\sin\theta)^2}$

$= \sqrt{1-\sin^2\theta}$

$= \sqrt{\cos^2\theta}$

$= \cos\theta$

3) Algebra to expand the square

4) Use power reducing identity on $\sin^2\theta$

EXAMPLE 3: Evaluate the definite integral.

$$\int_0^{\sqrt{3}/2} \frac{1}{\left(1-t^2\right)^{5/2}}\,dt$$

$$\int \frac{dt}{\left(\sqrt{1-t^2}\right)^5}$$

$$= \int \frac{\overset{1}{\cancel{\cos\theta}}\,d\theta}{(\cos\theta)^{\cancel{5}\,4}}$$

$$= \int \sec^4\theta\, d\theta$$

$$= \int \sec^2\theta \sec^2\theta\, d\theta$$

$$= \int (1+\tan^2\theta)\sec^2\theta\, d\theta$$

$$= \int \sec^2\theta\, d\theta + \int (\tan\theta)^2\,\underline{\sec^2\theta\, d}\theta$$

$$= \tan\theta + \frac{(\tan\theta)^3}{3} + C$$

$$= \frac{t}{\sqrt{1-t^2}} + \frac{1}{3}\cdot\left(\frac{t}{\sqrt{1-t^2}}\right)^3 + C$$

$$\int_0^{\sqrt{3}/2} \frac{dt}{(1-t^2)^{5/2}} = \left(\frac{t}{\sqrt{1-t^2}} + \frac{1}{3}\cdot\frac{t^3}{(1-t)^{3/2}}\right)\Bigg|_{t=0}^{t=\sqrt{3}/2}$$

$$= \left[\frac{\sqrt{3}/2}{\frac{1}{2}} + \frac{1}{3}\cdot\frac{\left(\frac{3^{1/2}}{2}\right)^3}{\left(\frac{1}{2}\right)^3}\right] - \left[\frac{0}{1} + \frac{1}{3}\cdot\frac{0}{1}\right]$$

Evil Plan

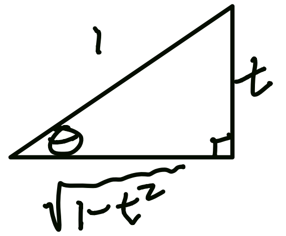
1) recognize that we have a square root with $a^2-(f(t))^2$ which is raised to a power

2) Trig Sub!

$a=1,\ f(t)=t$

$t = 1\sin\theta,\ \sin\theta = \frac{t}{1}$

$dt = \cos\theta\, d\theta$



$$\sqrt{1-t^2} = \sqrt{1-\sin^2 t}$$
$$= \sqrt{\cos^2 t}$$
$$= \cos t$$

3) trig integral

4) $g(\theta) = \tan\theta$
$g'(\theta) = \sec^2\theta$

$$= \sqrt{3} + \frac{1}{3} \cdot \frac{3^{3/2}}{\cancel{2^3}} \cdot \frac{\cancel{8}}{1}$$

$$= \sqrt{3} + \frac{1}{3} \cdot 3\sqrt{3}$$

$$= \boxed{2\sqrt{3}}$$

Section 8.5: Partial Fractions

When you finish your homework you should be able to…

- π Review how to decompose rational expressions into partial fractions
- π Utilize partial fractions to find indefinite integrals
- π Utilize partial fractions to evaluate definite integrals

Warm-up: Find the indefinite integral.

$$\int \frac{x^2-x-1}{x-1}dx = \int\left(x - \frac{1}{x-1}\right)dx$$

$$= \boxed{\frac{1}{2}x^2 - \ln|x-1| + C}$$

Evil Plan

Rewrite integrand by using long division

$$x - \frac{1}{x-1}$$

$$(x-1)\overline{)\,x^2 - x - 1}$$

$$\underline{-(x^2 - x)}\ \downarrow$$

$$-1$$

(CASE 1) Q HAS ONLY NONREAPEATED LINEAR FACTORS

Under the assumption that Q has only nonrepeated linear factors, the polynomial Q has the form

$$Q(x) = (x-a_1)(x-a_2)\cdots(x-a_n)$$

where no two of the numbers $a_1, a_2, \ldots, a_n$ are equal. In this case, the partial fraction decomposition of $\frac{P}{Q}$ is of the form

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \cdots + \frac{A_n}{x-a_n}$$

where the numbers $A_1, A_2, \ldots A_n$ are to be determined.

Example 1: Write the partial fraction decomposition of the rational expression in the integrand, and find the indefinite integral.

$$\int \frac{2}{9x^2-1}dx = \int \frac{2\,dx}{(3x-1)(3x+1)}$$

$$= \int \left(\frac{1 \cdot 3}{(3x-1) \cdot 3} - \frac{1 \cdot 3}{(3x+1) \cdot 3} \right) dx$$

$$= \frac{1}{3}\left(\ln|3x-1| - \ln|3x+1| \right) + C$$

$$= \frac{1}{3} \ln\left| \frac{3x-1}{3x+1} \right| + C$$

$$= \boxed{\ln\left| \sqrt[3]{\frac{3x-1}{3x+1}} \right| + C}$$

PFD

$$\frac{2}{(3x-1)(3x+1)} = \frac{\overset{\overset{1}{\uparrow}}{A_1}}{3x-1} + \frac{\overset{\overset{-1}{\uparrow}}{A_2}}{3x+1}$$

$$\frac{2}{(3x-1)(3x+1)} = \frac{A_1(3x+1) + A_2(3x-1)}{(3x+1)(3x-1)}$$

$$2 = 3A_1x + A_1 + 3A_2x - A_2$$

$$0x + 2x^0 = (3A_1 + 3A_2)x + (A_1 - A_2)x^0$$

$$3A_1 + 3A_2 = 0 \rightarrow A_1 = -A_2 = -(-1) = 1$$

$$A_1 - A_2 = 2 \rightarrow (-A_2) - A_2 = 2$$

$$-2A_2 = 2$$

$$A_2 = -1$$

Math 30
Intro Algebra
mathchick.net
6.5 video

Evil Plan

Rewrite integrand using PFD

(CASE 2) Q HAS REAPEATED LINEAR FACTORS

If the polynomial Q has a repeated linear factor, say $(x-a)^n$, $n \geq 2$, n is an integer, then, in the partial fraction decomposition of $\frac{P}{Q}$, we allow for the terms

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(x-a)^1} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_n}{(x-a)^n}$$

where the numbers $A_1, A_2, \ldots, A_n$ are to be determined.

Example 2: Write the partial fraction decomposition of the rational expression in the integrand, and find the indefinite integral.

$$\int \frac{5x-2}{(x-2)^2}\,dx = \int\left(\frac{5}{x-2} + \frac{8}{(x-2)^2}\right)dx$$

$$= 5\ln|x-2| + \frac{8(x-2)^{-1}}{-1} + C$$

$$= \boxed{\ln|(x-2)^5| - \frac{8}{x-2} + C}$$

Evil Plan

- we could do a basic u-sub and rewrite the 5x-2 in num. in terms of u.
- We'll use PFD

$$\frac{5x-2}{(x-2)^2} = \frac{\overset{5}{A_1}}{(x-2)^1} + \frac{\overset{8}{A_2}}{(x-2)^2}$$

$$\frac{5x-2}{(x-2)^2} = \frac{A_1(x-2) + A_2}{(x-2)^2}$$

$$5x-2 = A_1x - 2A_1 + A_2$$

$$5x - 2 = (A_1)x + (-2A_1 + A_2)$$

$$A_1 = 5$$

$$-2A_1 + A_2 = -2$$

$$\hookrightarrow -2(5) + A_2 = -2$$

$$A_2 = 8$$

(CASE 3) Q CONTAINS A NONREAPEATED IRREDUCIBLE QUADRATIC FACTOR

If Q contains a nonrepeated irreducible quadratic factor of the form $ax^2 + bx + c$, then, in the partial fraction decomposition of $\frac{P}{Q}$, allow for the term

$$\frac{P(x)}{Q(x)} = \frac{A_1x + B_1}{ax^2 + bx + c}$$

where the numbers A_1 and B_1 are to be determined.

Example 3: Write the partial fraction decomposition of the rational expression in the integrand, and find the indefinite integral.

$$\int \frac{6x}{x^3 - 8}dx = \int \frac{6x\,dx}{(x-2)(x^2+2x+4)}$$

$$= \int \left(\frac{1}{x-2} + \frac{-x+2}{x^2+2x+4}\right)dx$$

$g(x) = x^2 + 2x + 4$

$g'(x) = 2x + 2$

$g'(x) = 2(x+1)$

$$= \ln|x-2| - \int \frac{x-2+3-3}{x^2+2x+4}dx$$

$$= \ln|x-2| - \frac{1}{2}\int \frac{2(x+1)}{x^2+2x+4}dx + \int \frac{3\,dx}{x^2+2x+4}$$

$$= \ln|x-2| - \frac{1}{2}\ln|x^2+2x+4|$$

$$+ 3\int \frac{dx}{(\sqrt{3})^2 + (x+1)^2}$$

$$= \ln|x-2| - \ln\sqrt{|x^2+2x+4}$$

$$+ 3\left(\frac{1}{\sqrt{3}}\arctan\frac{x+1}{\sqrt{3}}\right) + C$$

Evil Plan

PFD

$$\frac{6x}{(x-2)(x^2+2x+4)} = \frac{\overset{1}{A_1}}{x-2} + \frac{\overset{-1}{A_2}x + \overset{2}{B_1}}{x^2+2x+4}$$

$$\frac{6x}{(x-2)(x^2+2x+4)} = \frac{A_1(x^2+2x+4) + (A_2x+B_1)(x-2)}{(x-2)(x^2+2x+4)}$$

$6x = A_1x^2 + 2A_1x + 4A_1 + A_2x^2 - 2A_2x + B_1x - 2B$

$0x^2 + 6x + 0 = (A_1 + A_2)x^2 + (2A_1 - 2A_2 + B_1)x + (4A_1 - 2B$

$A_1 + A_2 = 0 \rightarrow A_1 = -A_2 \rightarrow A_2 = -A$

$2A_1 - 2A_2 + B_1 = 6$

$4A_1 \quad - 2B_1 = 0 \rightarrow A_1 = \frac{1}{2}B_1$

$B_1 = 2A_1$

$2A_1 - 2(-A_1) + (2A_1) = 6$

$6A_1 = 6$

$A_1 = 1$

$A_2 = -1$

$B_1 = 2(1) = 2$

$$= \boxed{\ln\frac{|x-2|}{\sqrt{|x^2+2x+4|}} + \sqrt{3}\arctan\frac{x+1}{\sqrt{3}} + C}$$

PFD no

$$\frac{3}{x^2+2x+4} = \frac{A_1x+B_1}{x^2+2x+4}$$

Factor more
no

Complete the square yes

$$x^2+2x+4 = (x^2+2x+(1)^2)+4-1$$

$$= (x+1)^2+3$$

$$= (x+1)^2+(\sqrt{3})^2$$

$$= (\sqrt{3}) + (x+1)^2$$

(CASE 4) Q CONTAINS A REPEATED IRREDUCIBLE QUADRATIC FACTOR

If the polynomial Q contains a repeated irreducible quadratic factor of the form $(ax^2+bx+c)^n$, $n \geq 2$, n is an integer, then, in the partial fraction decomposition of $\frac{P}{Q}$, allow for the terms

$$\frac{P(x)}{Q(x)} = \frac{A_1x+B_1}{(ax^2+bx+c)^1} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

where the numbers $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ are to be determined.

Example 4: Write the partial fraction decomposition of the rational expression in the integrand, and evaluate the definite integral.

$$\int_0^1 \frac{x^3}{(x^2+16)^2}\,dx$$

$g(x) = x^2+16$

$g'(x) = 2x$

Evil plan

PFD

$$\frac{x^3}{(x^2+16)^2} = \frac{\overset{1}{A_1}x+\overset{0}{B_1}}{(x^2+16)^1} + \frac{\overset{-16}{A_2}x+\overset{0}{B_2}}{(x^2+16)^2}$$

(x^2+16)

$$\frac{x^3}{(x^2+16)^2} = \frac{A_1x^3+B_1x^2+16A_1x+16B_1+A_2x+B_2}{(x^2+16)^2}$$

$$1x^3+0x^2+0x+0 = (A_1)x^3+(B_1)x^2+(16A_1+A_2)x+(16B_1+B_2)$$

$A_1 = 1$

$B_1 = 0$

$16A_1+A_2 = 0 \rightarrow A_2 = -16$

$16B_1+B_2 = 0 \rightarrow B_2 = 0$

$$= \int_0^1 \left(\frac{2 \cdot 1x}{2(x^2+16)} - \frac{16}{2}\frac{x \cdot 2}{(x^2+16)^2}\right)dx$$

$$= \left[\frac{1}{2}\ln|x^2+16| - 8\frac{(x^2+16)^{-1}}{-1}\right]_{x=0}^{x=1}$$

$$= \left(\frac{1}{2}\ln 17 + \frac{8}{17}\right)$$

$$- \frac{1}{2}\ln 16 + \frac{8}{16}$$

$$= \frac{1}{2}\ln\frac{17}{16} + \frac{128-136}{272}$$

$$= \boxed{\ln\frac{\sqrt{17}}{4} - \frac{1}{34}}$$

Example 5: Find the indefinite integral.

$$\int \frac{5\cos x}{\sin^2 x + 3\sin x - 4}\,dx$$

$$= \int \frac{5\cos x\,dx}{u^2+3u-4}$$

$$= \int \frac{5\,du}{(u+4)(u-1)}$$

$$= \int \left(-\frac{1}{u+4} + \frac{1}{u-1} \right) du$$

$$= -\ln|u+4| + \ln|u-1| + C$$

$$= \ln\left|\frac{u-1}{u+4}\right| + C$$

$$= \ln\left|\frac{-1+\sin x}{4+\sin x}\right| + C$$

Evil Plan

1) let $u = \sin x$

$du = \cos x\,dx$

2) PFD

$$\frac{5}{(u+4)(u-1)} = \frac{\overset{-1}{A_1}}{u+4} + \frac{\overset{1}{A_2}}{u-1}$$

$$\frac{5}{(u+4)(u-1)} = \frac{A_1u - A_1 + A_2u + 4A_2}{(u+4)(u-1)}$$

$$0u + 5 = (A_1 + A_2)u + (-A_1 + 4A_2)$$

$$\cancel{A_1} + A_2 = 0$$

$$\underline{-\cancel{A_1} + 4A_2 = 5}$$

$$5A_2 = 5$$

$$A_2 = 1$$

$$A_1 + 1 = 0$$

$$A_1 = -1$$

Section 8.7: Indeterminate Forms and L'Hôpital's Rule

When you finish your homework you should be able to…

- π Recognize all indeterminate forms
- π Apply L'Hôpital's Rule to evaluate limits
- π Manipulate expressions so that L'Hôpital's Rule may be applied to evaluate limits

WARM-UP: Find the limit. It is okay to write ±∞ as your answer.

1. $\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9}$ $\overset{D.S.}{=}$ $\frac{\sqrt{9}-3}{9-9} = \frac{0}{0}$ indeterminate form... MORE work!! ◡

$$\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9}\cdot\frac{\sqrt{x}+3}{\sqrt{x}+3} = \lim_{x\to 9}\frac{\overset{1}{\cancel{x-9}}}{\underset{1}{\cancel{(x-9)}}(\sqrt{x}+3)}$$

$$= \lim_{x\to 9}\frac{1}{\sqrt{x}+3}$$

$$\overset{D.S.}{=} \frac{1}{\sqrt{9}+3} \longrightarrow = \boxed{\frac{1}{6}}$$

---

2. $\lim_{x\to 1}\frac{x^3-x^2-x+1}{x^2-1}$ $= \lim_{x\to 1}\frac{x^2(x-1)-1(x-1)}{(x-1)(x+1)}$

D.S. / $\frac{0}{0}$

$$= \lim_{x\to 1}\frac{(x^2-1)\overset{1}{\cancel{(x-1)}}}{\underset{1}{\cancel{(x-1)}}(x+1)}$$

$$= \lim_{x\to 1}\frac{\overset{1}{\cancel{(x+1)}}(x-1)}{\underset{1}{\cancel{x+1}}}$$

$$= \lim_{x\to 1}(x-1)$$

$$\overset{D.S.}{=} 1-1$$

$$= \boxed{0}$$

D.S. $\frac{0}{0}$

3. $$\lim_{\Delta x\to 0}\frac{\frac{x}{x}\cdot\frac{1}{(x+\Delta x)}-\frac{1}{x}\cdot\frac{(x+\Delta x)}{(x+\Delta x)}}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{x-(x+\Delta x)}{x(x+\Delta x)}\cdot\frac{1}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{-\cancel{\Delta x}}{x(x+\Delta x)}\cdot\frac{1}{\cancel{\Delta x}}$$

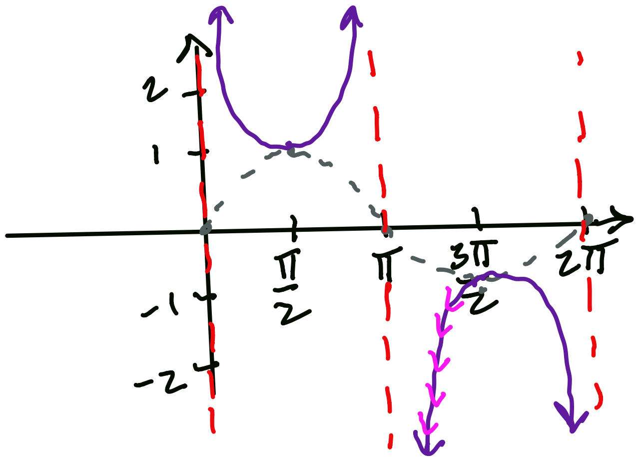
$$=\lim_{\Delta x\to 0}\frac{-1}{x(x+\Delta x)}$$

$$\overset{D.S.}{=}\frac{-1}{x(x+0)}$$

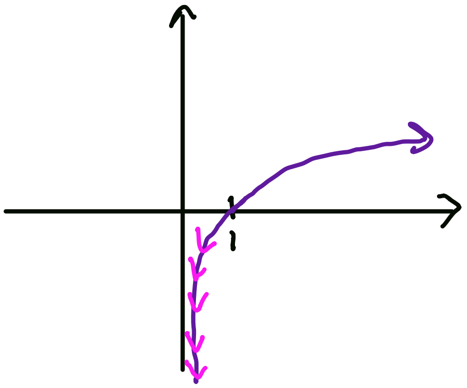
$$=\boxed{-\frac{1}{x^2}}$$

4. $\lim_{x\to\pi^+}\csc x \rightarrow$ DNE

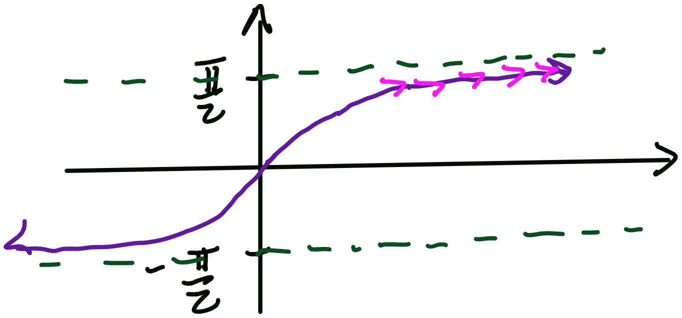
tends towards $\boxed{-\infty}$



5. $\lim\limits_{x\to 0^+} \ln x = \boxed{-\infty}$



6. $\lim\limits_{x\to\infty} \arctan x = \boxed{\frac{\pi}{2}}$



7. $^{4}\lim\limits_{x\to 0} \frac{\sin 4x}{4x} = 4\lim\limits_{x\to 0}\frac{\sin 4x}{4x}$

$= 4(1)$

special trig. limit

$= \boxed{4}$

D.S. $\frac{0}{0}$

8. $\lim\limits_{x\to 0} \frac{1-\cos^2 x}{x + x\cos x} = \lim\limits_{x\to 0}\frac{\overset{1}{\cancel{(1+\cos}}\ )(1-\cos x)}{x\,\underset{1}{\cancel{(1+\cos x)}}}$

Special trig limit: $\lim\limits_{\theta\to 0}\frac{1-\cos\theta}{\theta} = 0$

$= \lim\limits_{x\to 0}\frac{1-\cos x}{x}$

$= \boxed{0}$

special trig limit

What indeterminate form did you encounter in some of these problems?

$\frac{0}{0}$

What skills did you use to get these expressions into a determinate form?:

1. Algebra — mult by "1" — common denominator
   — rationalize numerator
   — factoring

2. Graphing

3. Special trig limits

L'Hôpital's Rule

Suppose f and g are differentiable and $g'(x) \neq 0$ near a (except possible at a).

Suppose that

$$\lim_{x\to a} f(x) = 0 \text{ and } \lim_{x\to a} g(x) = 0$$

or that

$$\lim_{x\to a} f(x) = \pm\infty \text{ and } \lim_{x\to a} g(x) = \pm\infty$$

meaning that we have an indeterminate form of $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$.

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

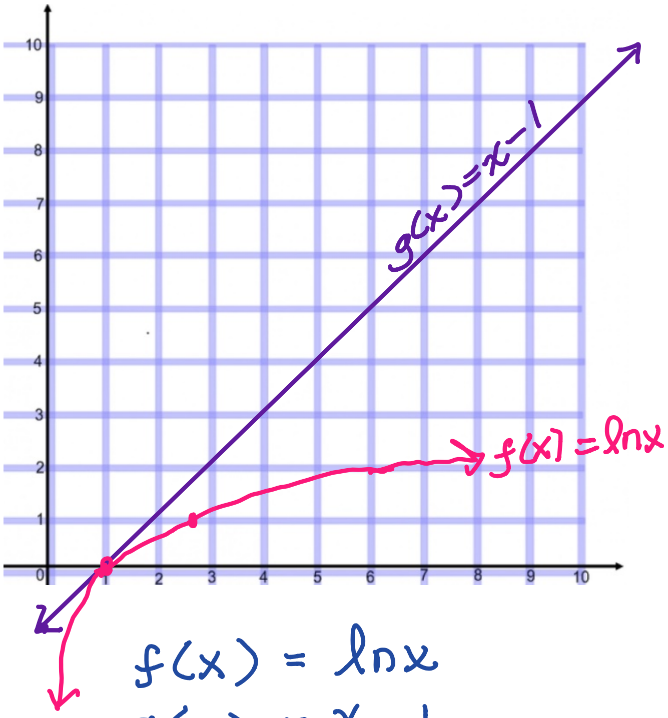
If the limit on the right side exists or is ∞ or $-\infty$.

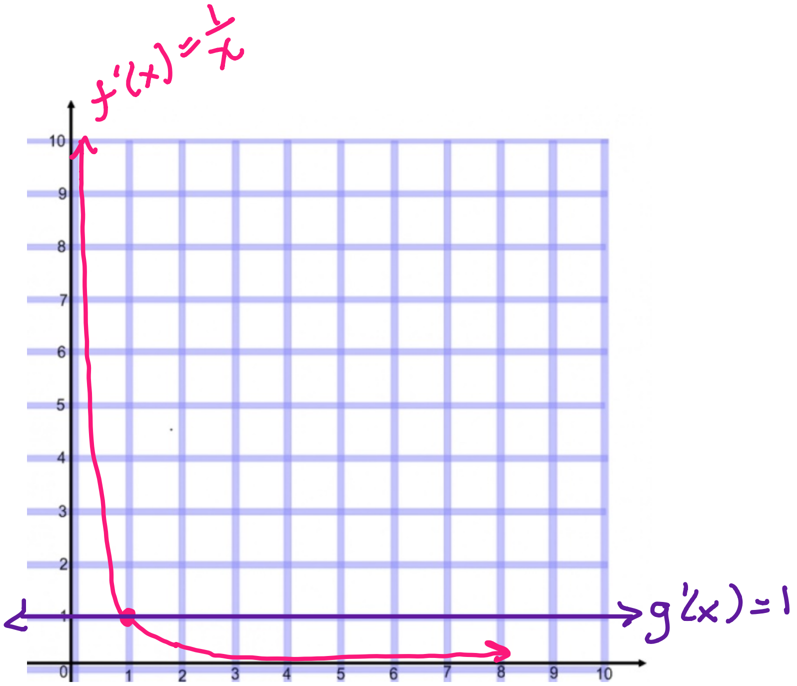
*What should we check before applying L'Hôpital's Rule?

1. f and g are differentiable near a and
2. $g'(x) \neq 0$ near a.

**L'Hôpital's Rule is also valid for one-sided limits and for limits at $-\infty$ or ∞.

***Let's look at the special case when $f(a)=g(a)=0$, f' and g' are continuous, and $g'(x)\neq 0$.





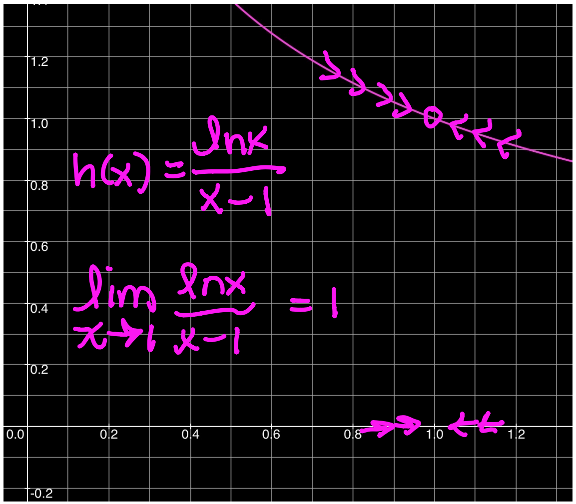
$f(x) = \ln x$

$g(x) = x - 1$

$a = 1$

$f'(x) = \frac{1}{x}$

$g'(x) = 1$



D.S. $\frac{0}{0}$

$$\lim_{x\to 1}\frac{\ln x}{x-1} \overset{LR}{=} \lim_{x\to 1}\frac{1/x}{1}$$

$$\overset{D.S}{=} \frac{1/1}{1}$$

$$= \boxed{1}$$

Example 1: Determine if L'Hôpital's Rule can be used to evaluate the limit. If so, apply L'Hôpital's Rule to evaluate the limit.

D.S. $\frac{0}{0}$

a. $\lim\limits_{x\to 9}\frac{\sqrt{x}-3}{x-9} \overset{LR}{=} \lim\limits_{x\to 9}\frac{\frac{d}{dx}(x^{1/2}-3)}{\frac{d}{dx}(x-9)}$

$= \lim\limits_{x\to 9}\frac{\frac{1}{2\sqrt{x}}}{1}$

$= \lim\limits_{x\to 9}\frac{1}{2\sqrt{x}}$

D.S.

$\overset{D.S.}{=} \frac{1}{2\sqrt{9}}$

$= \boxed{\frac{1}{6}}$

D.S. $\frac{0}{0}$

b. $\lim\limits_{x\to 1}\frac{x^3-x^2-x+1}{x^2-1} \overset{LR}{=} \lim\limits_{x\to 1}\frac{\frac{d}{dx}(x^3-x^2-x+1)}{\frac{d}{dx}(x^2-1)}$

$= \lim\limits_{x\to 1}\frac{3x^2-2x-1}{2x}$

$\overset{D.S.}{=} \frac{3(1)^2-2(1)-1}{2(1)}$

$= \frac{0}{2}$

$= \boxed{0}$

D.S. $\frac{0}{0}$

c. $\lim\limits_{x\to 0}\frac{\sin 4x}{x} \overset{LR}{=} \lim\limits_{x\to 0}\frac{4\cos 4x}{1}$

$\overset{D.S.}{=} 4\cos[4(0)]$

$= 4(1)$

$= \boxed{4}$

D.S.

$\frac{0}{0}$

d. $\lim\limits_{x\to 0}\frac{1-\cos^2 x}{x+x\cos x} \overset{LR}{=} \lim\limits_{x\to 0}\frac{-2(\cos x)^{1}(-\sin x)}{1+(1\cos x+x(-\sin x))}$

$= \lim\limits_{x\to 0}\frac{2\cos x\sin x}{1+\cos x-x\sin x}$

$\overset{D.S.}{=} \frac{2\cos 0\sin 0}{1+\cos 0-0\sin 0}$

$= \frac{0}{2}$

$= \boxed{0}$

D.S.

$\frac{\infty}{\infty}$

e. $\lim\limits_{x\to\infty}\frac{\ln x}{\sqrt{x}} \overset{LR}{=} \lim\limits_{x\to\infty}\frac{1/x}{1/2\sqrt{x}}$

$= \lim\limits_{x\to\infty}\frac{2\sqrt{x}}{x}$

$= \lim\limits_{x\to\infty}\frac{2}{\sqrt{x}}$

$= \boxed{0}$

Indeterminate Forms

We already know that $\frac{0}{0}$ and $\frac{\pm\infty}{\pm\infty}$ represent 2 types of indeterminate forms. There are also indeterminate products, differences, and powers.

Indeterminate Products occur when the limit of 1 factor approaches 0 and the other factor approaches ∞ or $-\infty$. Suppose $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=\infty$. If f prevails, the result of the limit of the product will be 0. If g is the victor, the limit of the product will be ∞. If they decide to sign a treaty, the answer will be some finite, real number. To find out, see if you can rewrite the difference into a quotient.

Example 2: Evaluate the limit.

D.S. $\infty\cdot 0$

a. $\lim_{x\to\infty}\sqrt{x}e^{-x/2} = \lim_{x\to\infty}\frac{\sqrt{x}}{e^{x/2}}$

D.S. $\frac{\infty}{\infty}$

$\overset{LR}{=} \lim_{x\to\infty}\frac{\frac{1}{2\sqrt{x}}}{\frac{1}{2}e^{x/2}}$

$= \lim_{x\to\infty}\frac{1}{\sqrt{x}e^{x/2}}$

$= \boxed{0}$

D.S. $0\cdot(-\infty)$

b. $\lim_{x\to 0^+}\sin x\ln x = \lim_{x\to 0^+}\frac{\ln x}{\csc x}$

D.S. $\frac{-\infty}{\infty}$

$\overset{LR}{=} \lim_{x\to 0^+}\frac{1/x}{-\csc x\cot x}$

$= \lim_{x\to 0^+}\frac{-1}{x\csc x\cot x}$

D.S. $\frac{0}{0}$

$= \lim_{x\to 0^+}\frac{-\sin x\tan x}{x}$

$\overset{LR}{=} \lim_{x\to 0^+}\frac{-\cos x\tan x-\sin x\sec^2 x}{1}$

$\overset{D.S.}{=} -\cos 0\tan 0-\sin 0(\sec 0)^2$

$= 0-0$

$= \boxed{0}$

Indeterminate Differences occur when both limits approach ∞.

Suppose $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$. If f prevails, the result of the limit of the difference will be ∞. If g is the victor, the limit of the ~~product~~ difference will be −∞. If they decide to sign a treaty, the answer will be some finite number. To find out, see if you can rewrite the difference into a quotient by using a common denominator, trig. identity, or factoring out a common factor.

Example 3: Evaluate the limit.

D.S. $\infty-\infty$

a. $\lim_{x\to 0^+}(\csc x - \cot x) = \lim_{x\to 0^+} \frac{1}{\sin x} - \frac{\cos x}{\sin x}$

$= \lim_{x\to 0^+} \frac{1-\cos x}{\sin x} \leftarrow$ D.S. $\frac{0}{0}$

$\overset{L'H}{=} \lim_{x\to 0^+} \frac{-(-\sin x)}{\cos x}$

$\overset{D.S.}{=} \frac{\sin 0}{\cos 0}$

$= \frac{0}{1}$

$\boxed{0}$

b. $\lim_{x\to 1^+}\left[\ln\left(x^7-1\right)-\ln\left(x^5-1\right)\right]$

Indeterminate Powers occur when $f(x)^{[g(x)]}$. There are 3 indeterminate forms that arise from this type of limit.

1. $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=0$. This yields the inderminate form 0^0.
2. $\lim_{x\to a} f(x)=\infty$ and $\lim_{x\to a} g(x)=0$. This yields the inderminate form ∞^0.
3. $\lim_{x\to a} f(x)=1$ and $\lim_{x\to a} g(x)=\infty$. This yields the inderminate form 1^∞.

To find these types of limits, see if you can take the natural logarithm:

$$\rightarrow \quad \ln M^p = p\ln M$$

or write the function as an exponential:

$$f(x)^{[g(x)]} = e^{\ln f(x)^{[g(x)]}} \qquad e^{\ln x}=x$$

Example 4: Evaluate the limit.

D.S. 1^∞

a. $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx}$

$$\ln y = \ln\left(1+\frac{a}{x}\right)^{bx}$$

$$\ln y = bx\ln\left(1+\frac{a}{x}\right) \quad \swarrow \infty\cdot 0$$

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty} bx\ln\left(1+\frac{a}{x}\right) \quad \swarrow \frac{0}{0}$$

$$= b\lim_{x\to\infty}\frac{\ln\left(1+\frac{a}{x}\right)}{\frac{1}{x}}$$

$$\overset{LR}{=} b\lim_{x\to\infty}\frac{\frac{1}{1+\frac{a}{x}}\cdot\left(\cancel{-}a\cancel{x^{-2}}\right)}{\cancel{-}\cancel{x^{-2}}}$$

$$= b\lim_{x\to\infty}\frac{a}{1+\frac{a}{x}}$$

D.S. 0^0

b. $\lim_{x\to 0^+} x^{\sqrt{x}}$

$$= \lim_{x\to 0^+} e^{\ln x^{\sqrt{x}}}$$

$$= \lim_{x\to 0^+} e^{\sqrt{x}\ln x} \quad \swarrow 0\cdot(-\infty)$$

$$= e^{\lim_{x\to 0^+}\sqrt{x}\ln x} \quad \swarrow -\frac{\infty}{\infty}$$

$$= e^{\lim_{x\to 0^+}\frac{\ln x}{1/\sqrt{x}}}$$

$$\overset{LR}{=} e^{\left(\lim_{x\to 0^+}\frac{1/x}{\frac{-1}{2x^{3/2}}}\right)}$$

$$= e^{\lim_{x\to 0^+} -2x^{1/2}}$$

$$= e^0 \rightarrow = \boxed{1}$$

$$= ab \lim_{x\to\infty} \frac{1}{1+\frac{a}{x}}$$

$$= ab(1)$$

$$= ab$$

Now,

$$\ln y = ab$$

$$e^{ab} = y$$

$$\lim_{x\to\infty} \left(1+\frac{a}{x}\right)^{bx} = \boxed{e^{ab}}$$

Section 8.8: Improper Integrals

When you finish your homework you should be able to…

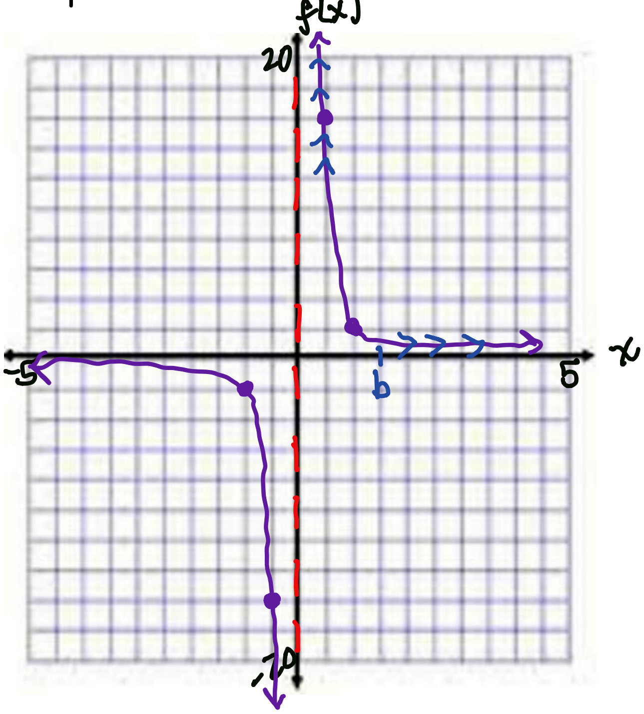
- π Recognize when a definite integral is improper
- π Use your integration and limit techniques to evaluate improper integrals

WARM-UP: Consider the function $f(x)=\dfrac{2}{x^3}$.

$\int \frac{2}{x^3}dx$ is fine, but

$\int_0^1 \frac{2}{x^3}dx \rightarrow$ there's a VA at $x=0$

1. Graph the function.



2. Find the limits. It is okay to write ±∞ as your answer.

a. $\lim_{x\to 0^+}\frac{2}{x^3} = \infty$

b. $\lim_{x\to\infty}\frac{2}{x^3} = 0$

3. Evaluate the definite integral.

$$\int_1^b \frac{2}{x^3}dx = 2\int_1^b x^{-3}dx$$

$$= \cancel{2}\frac{x^{-2}}{-\cancel{2}}\Big|_{x=1}^{x=b}$$

$$= -\left(\frac{1}{b^2}-\frac{1}{1}\right)$$

$$= 1-\frac{1}{b^2}$$

Let's put some stuff together☺

Recall the if a function is nonnegative on the interval $[a,b]$, the definite integral is equal to the area under the curve and bounded by the x-axis, $x=a$, $x=b$. Also remember that a function is said to have an infinite discontinuity at c when, from the right or the left,

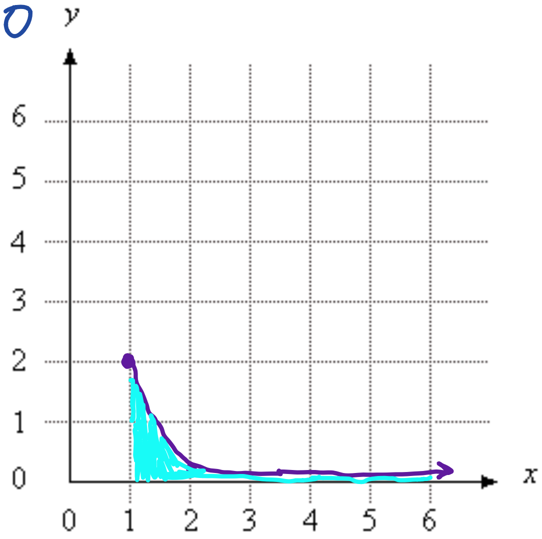
$$\lim_{x\to c^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x\to c^-} f(x) = \pm\infty$$

TYPE 1: INFINITE INTERVALS

Now consider the following definite integral.

$$\int_1^\infty \frac{2}{x^3}\,dx = \lim_{b\to\infty}\int_1^b \frac{2}{x^3}\,dx = \lim_{b\to\infty}\left(1-\frac{1}{b^2}\right) = 1-0$$

$$\boxed{1}$$



Definition: Improper Integrals With Infinite Integration Limits

1. Suppose f is continuous on the interval from $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

2. Suppose f is continuous on the interval from $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$$

3. Suppose f is continuous on the interval from ___________, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx = \underbrace{\lim_{a\to-\infty} \int_a^c f(x)\,dx}_{} + \underbrace{\lim_{b\to\infty} \int_c^b f(x)\,dx}_{}$$

and/or

where c is any real number.

In the first two cases, the improper integral converges when the limit exists; otherwise the improper integral diverges. In the third case, the improper integral on the left diverges when either of the improper integrals on the right diverge.

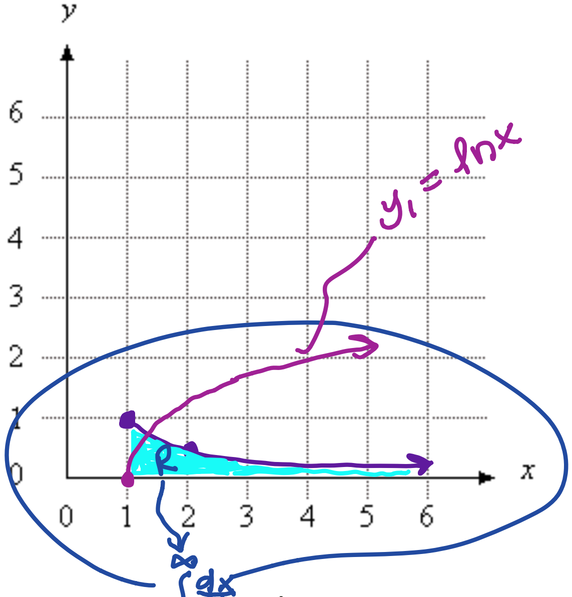
Example 1: If possible, evaluate the following definite integrals and ascertain if they are convergent or divergent.

a. $\int_1^\infty \frac{dx}{x}$ $= \lim_{b\to\infty} \int_1^b x^{-1}\,dx$

$= \lim_{b\to\infty} \ln|x| \Big|_{x=1}^{x=b}$

$= \lim_{b\to\infty} \left(\underbrace{\ln|b|}_{\infty} - \overbrace{\ln|1|}^{0}\right)$

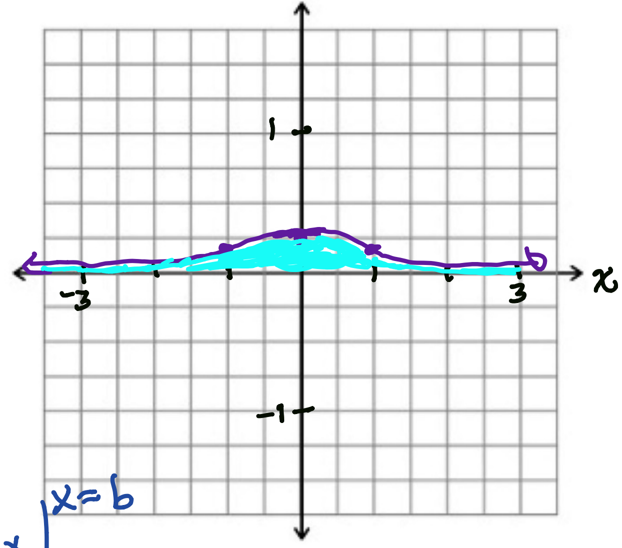
$\hookrightarrow \infty$, $\boxed{\text{So } \int_1^\infty \frac{dx}{x} \text{ diverges}}$



b. $\int_{-\infty}^\infty \frac{dx}{4+x^2}$

$= \int_{-\infty}^0 \frac{dx}{4+x^2} + \int_0^\infty \frac{dx}{4+x^2}$

$= \lim_{a\to-\infty} \int_a^0 \frac{dx}{2^2+x^2} + \lim_{b\to\infty} \int_0^b \frac{dx}{2^2+x^2}$



$= \lim_{a\to-\infty} \frac{1}{2}\arctan\frac{x}{2}\Big|_{x=a}^{x=0} + \lim_{b\to\infty} \frac{1}{2}\arctan\frac{x}{2}\Big|_{x=0}^{x=b}$

$= \frac{1}{2}\left[\left(\arctan\frac{0}{2} - \lim_{a\to-\infty}\arctan\frac{a}{2}\right) + \left(\lim_{b\to\infty}\arctan\frac{b}{2} - \arctan\frac{0}{2}\right)\right]$

$= \frac{1}{2}\left[\left(0 - \left(-\frac{\pi}{2}\right)\right) + \left(\frac{\pi}{2} - 0\right)\right]$

$= \frac{1}{2}(\pi)$

$= \frac{\pi}{2}$ $\boxed{\text{So, } \int_{-\infty}^\infty \frac{dx}{4+x^2} \text{ converges to } \frac{\pi}{2}.}$

TYPE 2: DISCONTINUOUS INTEGRANDS

Now consider the following definite integral.

$$\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx$$

Evil Plan:

IBP

$u = \ln x \qquad \int dv = \int x^{-1/2}\,dx$

$du = \frac{1}{x}\,dx \qquad v = 2x^{1/2}$

Consider: $\int x^{-1/2} \ln x \, dx$

$$= 2x^{1/2}\ln x - 2\int x^{1/2} \cdot \frac{1}{x}\,dx$$

$$= 2x^{1/2}\ln x - 2\int x^{-1/2}\,dx$$

$$= 2x^{1/2}\ln x - 2 \cdot 2x^{1/2} + C$$

$$= 2\sqrt{x}\,(\ln x - 2) + C$$

$$\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx = \lim_{a\to 0^+} \int_a^1 \frac{\ln x}{\sqrt{x}}\,dx$$

$$= \lim_{a\to 0^+} 2\sqrt{x}\,(\ln x - 2)\Big|_{x=a}^{x=1}$$

$$= 2\left[\lim_{a\to 0^+} \sqrt{1}\,(\overset{0}{\widetilde{\ln 1}} - 2) - \boxed{\lim_{a\to 0^+} \sqrt{a}\,(\ln a - 2)}\right]$$

$$= 2\left[(-2) - \overset{0\,(-\infty)}{\boxed{(\sqrt{0}\,(-\infty - 2))}}\right] \text{ D.S.}$$

$$\lim_{a\to 0^+} \sqrt{a}\,(\ln a - 2)$$

$$= \lim_{a\to 0^+} \frac{(\ln a) - 2}{\frac{1}{\sqrt{a}}}$$

$$\overset{LR}{=} \lim_{a\to 0^+} \frac{1/a}{-\frac{1}{2}a^{-3/2}}$$

$$= \lim_{a\to 0^+} -2a^{1/2}$$

$$= 0$$

$$= 2(-2 - 0)$$

$$= -4$$

So, $\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx$ converges to -4

Definition: Improper Integrals With Infinite Discontinuities

1. Suppose f is continuous on the interval from (a, b], and has an infinite discontinuity at a, then

$$\int_a^b f(x)\,dx = \lim_{a \to c^+} \int_c^b f(x)\,dx$$

2. Suppose f is continuous on the interval from [a, b), and has an infinite discontinuity at b, then

$$\int_a^b f(x)\,dx = \lim_{b \to c^-} \int_a^c f(x)\,dx$$

3. Suppose f is continuous on the interval from [a, b], except for some c in (a, b) at which f has an infinite discontinuity at c, then

$$\boxed{\int_a^b f(x)\,dx =} \lim_{b \to c^-} \int_a^c f(x)\,dx + \lim_{a \to c^+} \int_c^b f(x)\,dx$$

In the first two cases, the improper integral converges when the limit exists; otherwise the improper integral diverges. In the third case, the improper integral on the left diverges when either of the improper integrals on the right diverge.

Example 2: If possible, evaluate the following definite integrals and ascertain if they are convergent or divergent.

a. $\int_3^6 \frac{dx}{\sqrt{36-x^2}}$ $= \lim_{b \to 6^-} \int_3^b \frac{dx}{\sqrt{6^2-x^2}}$

$$= \lim_{b \to 6^-} \arcsin \frac{x}{6} \Big|_{x=3}^{x=b}$$

$$= \lim_{b \to 6^-} \left(\arcsin \frac{b}{6} - \arcsin \frac{3}{6} \right)$$

D.S.

$$= \arcsin 1 - \arcsin \frac{1}{2} \rightarrow = \frac{\pi}{2} - \frac{\pi}{6} \rightarrow = \frac{\pi}{3}$$

So, $\int_3^6 \frac{dx}{\sqrt{36-x^2}}$ converges to $\frac{\pi}{3}$

b. $\int_1^\infty \frac{dx}{x \ln x}$

Consider $\int \frac{dx}{x^p}$, where *p* is a real number. Let's find the indefinite integral on

1. $p = 0$
2. $p \neq 1$
3. $p = 1$

Example 3: Determine all values of *p* for which the improper integral converges.

$$\int_1^\infty \frac{dx}{x^p}$$

THEOREM: A SPECIAL TYPE OF IMPROPER INTEGRAL

$$\int_1^\infty \frac{dx}{x^p} = \begin{cases} \dfrac{1}{p-1}, & p > 1 \\ \text{diverges}, & p < 1 \end{cases}$$

Example 4: If possible, evaluate the following definite integrals and ascertain if they are convergent or divergent.

a. $\int_1^\infty \frac{dx}{x^{1/2}}$

b. $\int_1^\infty x^{-3}dx$

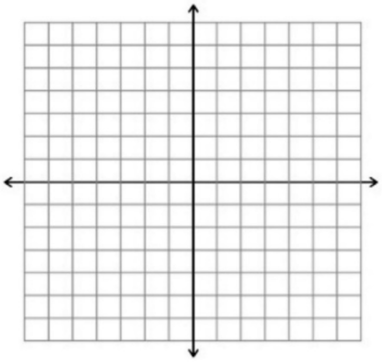
9.1: Sequences

When you finish your homework you should be able to…

π Identify the terms of a sequence, write a formula for the *n*th term of a sequence, and ascertain whether a sequence converges or diverges.
π Use properties of monotonic sequences and bounded sequences.

WARM-UP: Consider the function $f(x)=\sqrt{x}$.

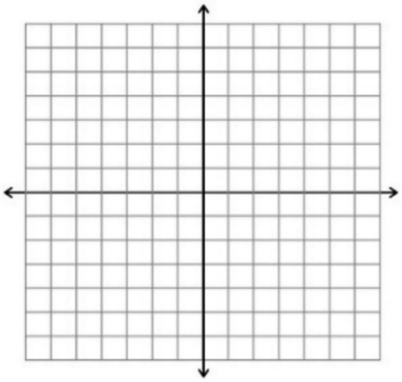
1. Sketch the graph of the function.



2. Find the following:
   a. $f(0)=$ ______
   b. $f(1)=$ ______
   c. $f(2)=$ ______
   d. $f(3)=$ ______
   e. $f(4)=$ ______
   f. $f(5)=$ ______
   g. $\lim\limits_{x\to 0}\sqrt{x}=$ ________
   h. $\lim\limits_{x\to 1}\sqrt{x}=$ ________
   i. $\lim\limits_{x\to 4}\sqrt{x}=$ ________
   j. $\lim\limits_{x\to \infty}\sqrt{x}=$ ________
3. Now consider $a_n=\sqrt{n}$.
4. Find the following:
   a. $a_1=$ ______
   b. $a_2=$ ______
   c. $a_3=$ ______
   d. $a_4=$ ______
   e. $a_5=$ ______

Hmmm…so it looks like ________ equals ________ at all of the ______________ ______________.

5. Sketch the graph of the sequence.



Definition of the Limit of a Sequence

Let *L* be a real number. The _______________ of a sequence _______ is _____, written as

$$\lim_{n \to \infty} a_n = L$$

if for $\varepsilon > 0$, there exists $M > 0$ such that $|a_n - L| < \varepsilon$ whenever $n > M$. If the limit *L* exists, then the sequence __________________. If the limit does not exist, then the sequence __________________.

Looking at the two graphs we sketched, as ___________ it looks like ___________. So, we would say the $\lim_{n \to \infty} a_n$ ________. and ________ ___________.

EXAMPLE 1: Write the first five terms of the sequence.

a. $a_n = \frac{3n}{n+4}$

b. $a_1 = 6,\ a_{k+1} = \frac{1}{3}a_k{}^2$

FACTORIALS are factors which decrease by one. So 5!, read as "five factorial" is

5! = ___________________ = _______. We will be working with unknown factorials.

In general, *n*! = ___________________________________, and 0! = _____.

EXAMPLE 2: Simplify the ratio of factorials.

a. $\frac{n!}{(n+2)!}$

b. $\frac{(2n+2)!}{(2n)!}$

EXAMPLE 3: Find the *n*th term of the sequence.

a. $1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \ldots$

b. $\frac{1}{2\cdot3}, \frac{1}{3\cdot4}, \frac{1}{4\cdot5}, \frac{1}{5\cdot6}, \ldots$

Theorem: Limit of a Sequence

Let *L* be a real number. Let f be a function of a real variable such that

If $\{a_n\}$ is a sequence such that $f(n) = a_n$ for every positive integer *n*, then

EXAMPLE 4: Find the limit of the sequence, if it exists.

a. $a_n = 6 + \frac{2}{n^2}$

b. $a_n = \cos\frac{2}{n}$

Theorem: Properties of Limits of Sequences

Let $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = K$.

1. $\lim_{n\to\infty}\left(a_n \pm b_n\right) =$ ______________
2. $\lim_{n\to\infty} ca_n =$ _____, c is any _________ number.
3. $\lim_{n\to\infty}\left(a_n b_n\right) =$ ______________
4. $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) =$ _____, ___________ and _________.

EXAMPLE 5: Determine the convergence or divergence of the sequence with the given *n*th term. If the sequence converges, find its limit.

a. $a_n = \frac{1+\left(-1\right)^n}{n^2}$

b. $a_n = \frac{\sqrt[3]{n}}{\sqrt[3]{n}+1}$

c. $a_n = \frac{(n-2)!}{n!}$

Absolute Value Theorem

For the sequence $\{a_n\}$, if

$\lim_{n\to\infty}|a_n| = 0$ then _____________________.

Squeeze Theorem for Sequences

If $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} b_n$ and there exists an integer N such that___________________,

then _________________________.

EXAMPLE 6: Show that the sequence converges and find its limit.

$$c_n = (-1)^n \frac{1}{n!}$$

Definition: Monotonic Sequence

A sequence $\{a_n\}$ is ________________ when its terms are ________________, $a_1 \le a_2 \le a_3 \le \cdots \le a_n \le \cdots$ or when its terms are nonincreasing ______________________________.

Definition: Bounded Sequence

1. A sequence $\{a_n\}$ is ________________ above when there is a real number ____ such that $a_n \le M$ for all ____. The number ____ is called an __________ ____________ of the sequence.
2. A sequence $\{a_n\}$ is bounded ____________ when there is a real number ____ such that $N \le a_n$ for all ____. The number ____ is called a __________ bound of the sequence.
3. A sequence _____ is _______________ when it is bounded ____________ and ____________ below.

Theorem: Bounded Monotonic Sequences

If a sequence $\{a_n\}$ is _________________ and ______________________, it ____________________.

EXAMPLE 7: Determine whether the sequence with the given *n*th term is monotonic and whether it is bounded.

$$a_n = \frac{\cos n}{n}$$

EXAMPLE 8: Fibonacci posed the following problem: Suppose that rabbits live forever and that every month each pair produces a new pair which becomes productive at age 2 months. If we start with one newborn pair, how many pairs of rabbits will we have in the *n*th month?

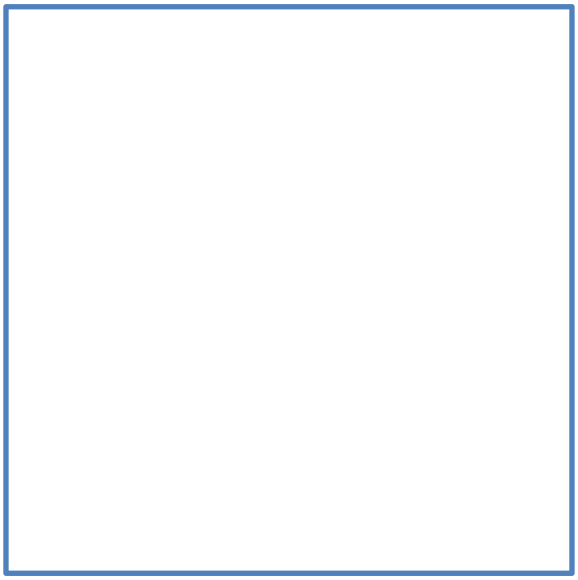
9.2: Series and Convergence

When you finish your homework you should be able to…

- π Understand and represent a convergent infinite series.
- π Use properties of infinite geometric series.
- π Use the *n*th term test for **divergence**.

We spent the last section checking out ___________________, and ascertaining whether a given sequence , a_n , __________________ or ___________________ as ________________. Remember, the _____________ of a sequence are represented as a ______ or _________, which need not be ordered. There are finite and ________________ sequences. What if we were interested in ________________ a sequence? If we are interested in summing a finite number, say *n*, of the ______________ of a sequence, we would be finding the _____ _________________ _______. If we are interesting in finding the sum of an infinite sequence, if it exists, we would be finding an ______________ sum, called an infinite ____________, or just a ____________.

Our main interest will be to ascertain whether a series ________________ or ________________.



EXAMPLE 1: Consider the sequence we found above.

a. Write the first five terms, and the *n*th term of the sequence.

b. Sum the first five terms.

c. Represent this 5th partial sum as a summation.

d. Find the limit of the sequence.

e. Find an expression for the *n*th partial sum.

f. What must the limit of this expression equal?

Definition: Convergent and Divergent Series

For the infinite series $\sum_{n=1}^{\infty} a_n$, The ____ ____________ sum is

If the sequence of partial sums $\{S_n\}$ _______________ to S , then the series $\sum_{n=1}^{\infty} a_n$ converges. The limit S is called the __________ of the _____________.

If $\{S_n\}$ diverges, then the series ____________________.

So from our first example, $S =$ _______________ , and this series

___________________ since the ____ _______________ sum _____________.

GEOMETRIC SERIES:

Theorem: Convergence of a Geometric Series

A geometric series with _________ ___ converges to the sum

when _____________________. Otherwise, for _____________, the series diverges.

EXAMPLE 2: Express the number as a ratio of integers.

$0.\overline{46}$

EXAMPLE 3: Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$ is convergent and find its sum.

NOTE: The series in example 3 is called a ________________________ series.

EXAMPLE 4: Show that the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

NOTE: The series in example 4 is called a ________________________ series.

Theorem: Properties of Infinite Series

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series, and let A, B, and c be real numbers.

If $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$, then the following series converge to the indicate sums.

1. $\sum_{n=1}^{\infty} ca_n =$ ___
2. $\sum_{n=1}^{\infty} (a_n + b_n) =$ ____
3. $\sum_{n=1}^{\infty} (a_n - b_n) =$ ____

EXAMPLE 5: Determine the convergence or divergence of the series. If the series converges, find its sum.

$$\sum_{n=1}^{\infty}\frac{1+2^n}{3^n}$$

Theorem: Limit of the *n*th Term of a Convergent Series

If $\sum_{n=1}^{\infty} a_n$ converges, then ____________________.

Theorem: *n*th Term Test for Divergence

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ ________________________.

EXAMPLE 6: Determine the convergence or divergence of the series. Explain.

a. $\sum_{n=1}^{\infty} \arctan n$

b. $\displaystyle\sum_{n=1}^{\infty}\frac{(-3)^{n-1}}{4^{n}}$

c. $\displaystyle\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n}$

EXAMPLE 7: Find all values of x for which the series converges. For these values of x, write the sum as a function of x.

$$\sum_{n=0}^{\infty} 5\left(\frac{x-2}{3}\right)^n$$

EXAMPLE 8: A ball is dropped from a height of 16 feet. Each time it drops h feet, it rebounds $0.81h$ feet. Find the total distance traveled by the ball.

9.3: The Integral Test, P-Series, and Harmonic Series

When you finish your homework you should be able to…

- π Use the Integral Test to ascertain whether an infinite series converges or diverges.
- π Determine whether a *p*-series converges or diverges.
- π Use properties of harmonic series.

WARM-UP: Determine whether the improper integral converges or diverges.

1. $\int_1^\infty \frac{\ln x}{x^3}\,dx$

2. $\int_1^{\infty} \frac{1}{3^x} dx$

3. $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$

Theorem: The Integral Test

If f is _______________, ________________, and __________________ for $x \geq 1$ and $a_n = f(n)$, then

Either both ______________________ or both ______________________.

*****NOTE:** Our interest is whether the series converges or diverges as ________, so the index of the summation can start at some integer _____ as opposed to a___ when we apply the integral test.

EXAMPLE 1: Determine the convergence or divergence of the series. Explain.

a. $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

b. $\sum_{n=1}^{\infty} \frac{n}{n^4 + 2n^2 + 1}$

P-Series and Harmonic Series

A harmonic series is the ____________ of sounds represented by _____________ waves in which the _______________ of each sound is an ______________ multiple of the _______________ frequency. Pythagoras and his students discovered this relationship between the _______ and the _________ of the vibrating string. The most beautiful harmonies seemed to correspond with the simplest ____________ of _________ numbers. Later mathematicians developed this idea into the ______________ series, where the ___________ in the harmonic series correspond to the node on a _________________ string that produce _________________ of the fundamental frequency. So, ___________ is ______________ the fundamental frequency, ___________ is ______________ times the fundamental frequency, and so on. In music, strings of the same ________________, ____________________, and ________________, and whose _________________ form a harmonic series, produce _______________ tones. A general harmonic series is of the form ___________________________________.

The harmonic series is a special case of the ______________, where __________.

Theorem: Convergence of *p*-Series

The *p*-series

________________ for ________, and ________________ for ______________.

EXAMPLE 2: Determine the convergence or divergence of the series. Explain.

a. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

b. $1 + \frac{1}{\sqrt[5]{4}} + \frac{1}{\sqrt[5]{9}} + \frac{1}{\sqrt[5]{16}} + \frac{1}{\sqrt[5]{25}} + \cdots$

9.4: Series Comparison Tests

When you finish your homework you should be able to…

- π Use the Direct Comparison Test to ascertain whether an infinite series converges or diverges.
- π Use the Limit Comparison Test to ascertain whether an infinite series converges or diverges.

WARM-UP: Determine whether the series converges or diverges.

1. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n+1}}$

2. $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$

Our Tests So Far…

nth Term Test for ____________________. If ______________, the series ___________________. If _________________, we need to ______ further __________!!!

Geometric Series is of the form ______________. If _________________, the series __________________ and its _______ is _______________________. Otherwise, the series diverges.

Telescoping Series. Requires __________________ ________________ decomposition. The ___________ is the sum of the terms which do not ______________ out plus ___________________.

***p*-Series** is of the form __________________. If _____________, the series ______________________. If ______________________, the series ____________________.

The Integral Test requires that ____ is ________________, continuous, and __________________ for $x \geq 1$, and $f(n) = a_n$ for all n. If ______________ converges, _______________ converges. Otherwise, ____________ diverges.

Theorem: The Direct Comparison Test

Let ____________________ for all n.

1. If $\sum_{i=1}^{\infty} b_n$ ____________________, ______________ converges.
2. If $\sum_{i=1}^{\infty} a_n$ ____________________, $\sum_{i=1}^{\infty} b_n$ ___________________.

EXAMPLE 1: Determine the convergence or divergence of the series. Explain.

a. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$

b. $\sum_{n=1}^{\infty} \frac{3^n}{2^n - 1}$

Theorem: The Limit Comparison Test

If _________, _________, and

where L is ____________ and ______________, then

$\sum_{i=1}^{\infty} a_n$ and $\sum_{i=1}^{\infty} b_n$ either both __________________ or both ________________.

NOTE: When choosing your comparison, you can disregard all but the

_________________ powers of _____. So, if we are testing $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{5n^2+2}$, our

comparison series would be _____________ = ___________.

Proof:

EXAMPLE 2: Determine the convergence or divergence of the series. Explain.

a. $\sum_{n=1}^{\infty} \frac{n}{n^4 + 2n^2 + 1}$

b. $\sum_{n=0}^{\infty} \frac{1+\sin n}{10^n}$

9.5: Alternating Series

When you finish your homework you should be able to…

- π Use the Alternating Series Test to ascertain whether an infinite series converges or diverges.
- π Use the Alternating Series Remainder to approximate the sum of an alternating series.
- π Classify a convergent series as conditionally convergent or absolutely convergent.

WARM-UP: Determine whether the series converges or diverges.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2)^{n+1}}$$

Theorem: Alternating Series Test

Let ______________________. The alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ and $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$ converge when both conditions below are met.

1. ______________________
2. ______________________, for all n.

NOTE: The second condition can be modified to require that ________________ for all ____ greater than some integer _____.

EXAMPLE 1: Determine the convergence or divergence of the series. Explain.

a. $\sum_{n=0}^{\infty}\frac{(-1)^n}{(2)^{n+1}}$

b. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$

c. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^2+4}$

d. $\sum_{n=1}^{\infty} \frac{1}{n} \cos n\pi$

e. $$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1\cdot3\cdot5\cdot7\cdots(2n-1)}{1\cdot4\cdot7\cdot10\cdots(3n-2)}$$

Theorem: Alternating Series Remainder

If a convergent alternating series satisfies the condition $a_{n+1} \le a_n$, then the

_________________ value of the ________________ _______ involved in

approximating the sum _____ by _____ is less than or equal to the first

_________________ term.

EXAMPLE 2: Approximate the sum of the series by using the first six terms.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{3^n}$$

EXAMPLE 3: Determine the number of terms required to approximate the sum of the series with an error of less than 0.001.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}$$

Theorem: Absolute Convergence

If the series __________ converges, then the series _________ also converges.

Which of our examples would be an example of this theorem?

Definition of Absolute and Conditional Convergence

1. The series $\sum a_n$ is ______________________ convergent when __________ converges.

2. The series $\sum a_n$ is ______________________ convergent when ________ converges but __________ diverges.

EXAMPLE 4: Determine whether the series converges absolutely or conditionally, or diverges.

a. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{e^{n^2}}$

b. $\sum_{n=1}^{\infty}(-1)^{n+1}\arctan n$

c. $$\sum_{n=1}^{\infty}\frac{\sin\left[\left(2n+1\right)\frac{\pi}{2}\right]}{n}$$

9.6: The Ratio and Root Tests

When you finish your homework you should be able to…

- π Use the Ratio Test to ascertain whether an infinite series converges or diverges.
- π Use the Root Test to ascertain whether an infinite series converges or diverges.
- π Review Tests for convergence and divergence of an infinite series.

Theorem: The Ratio Test

Let $\sum a_n$ be a series with ____________________ terms.

1. The series $\sum a_n$ converges ________________ when ____________________.
2. The series $\sum a_n$ diverges when _________________ or _________________.
3. The Ratio Test is _________________ when __________________.

EXAMPLE 1: Determine the convergence or divergence of the series using the Ratio Test.

a. $\sum_{n=0}^{\infty}\left(\frac{2}{e}\right)^n$

b. $\displaystyle\sum_{n=0}^{\infty}\frac{2^n}{n!}$

c. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(n+2)}{n(n+1)}$

d. $\sum_{n=0}^{\infty}\frac{\left(n!\right)^{2}}{\left(3n\right)!}$

Theorem: The Root Test

1. The series $\sum a_n$ converges ________________ when ____________________.

2. The series $\sum a_n$ diverges when _________________ or _________________.

3. The Root Test is _________________ when __________________.

EXAMPLE 2: Determine the convergence or divergence of the series using the Root Test.

a. $\sum_{n=1}^{\infty} \frac{1}{n^n}$

b. $\sum_{n=1}^{\infty}\left(\frac{n-2}{5n+1}\right)^{n}$

c. $\sum_{n=1}^{\infty} \left(\frac{\ln n}{n} \right)^{n}$

d. $\sum_{n=1}^{\infty} \frac{(n!)^n}{(n^n)^2}$

NOW IT'S UP TO YOU!!! DETERMINE WHETHER THE FOLLOWING INFINITE SERIES CONVERGE OR DIVERGE

1. $\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdot 7\cdots(2n-1)}{2\cdot 5\cdot 8\cdot 11\cdots(3n-1)}$

Step 1: Identify the test(s) and conditions (if applicable).

Step 2: Run the test.

Step 3: Conclusion.

2. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n\sqrt{n}}{n+1}$

Step 1: Identify the test(s) and conditions (if applicable).

Step 2: Run the test.

Step 3: Conclusion.

3. $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3 + 3n}}$

Step 1: Identify the test(s) and conditions (if applicable).

Step 2: Run the test.

Step 3: Conclusion.

4. $\displaystyle\sum_{n=1}^{\infty}\frac{3\cdot5\cdot7\cdots(2n+1)}{18^n n!(2n-1)}$

Step 1: Identify the test(s) and conditions (if applicable).

Step 2: Run the test.

Step 3: Conclusion.

5. $\sum_{n=1}^{\infty}\left(\frac{n}{500}\right)^n$

Step 1: Identify the test(s) and conditions (if applicable).

Step 2: Run the test.

Step 3: Conclusion.

6. $\sum_{n=1}^{\infty} e^{-4n}$

Step 1: Identify the test(s) and conditions (if applicable).

Step 2: Run the test.

Step 3: Conclusion.

7. $\sum_{n=1}^{\infty} \frac{5^n - 1}{6^n - 1}$

Step 1: Identify the test(s) and conditions (if applicable).

Step 2: Run the test.

Step 3: Conclusion.

8. $\sum_{n=1}^{\infty} \arctan n$

Step 1: Identify the test(s) and conditions (if applicable).

Step 2: Run the test.

Step 3: Conclusion.

9. $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

Step 1: Identify the test(s) and conditions (if applicable).

Step 2: Run the test.

Step 3: Conclusion.

10. $$\sum_{n=1}^{\infty} \frac{2^n}{4n^2 - 1}$$

Step 1: Identify the test(s) and conditions (if applicable).

Step 2: Run the test.

Step 3: Conclusion.

9.7: Taylor Polynomials

When you finish your homework you should be able to…

- π Find Taylor and Maclaurin polynomial approximations of elementary functions.
- π Use the remainder of a Taylor polynomial.

Some uses of the Taylor series for analytic functions include:

- The _____________ ________ of the series can be used as ______________________ of the entire function. Keep in mind that you need a sufficient amount of ______________.
- ______________________ and ______________________ of power series is _____________________ since it can be done ____________ by term.
- ___________________ operations can be done on the _______________ series ______________________. For example, ________________ formula follows from Taylor series __________________ for _________________ and _____________________ functions. This result is important in the field of _________________ analysis.
- _______________________ using the first few terms of a Taylor series can make otherwise _______________ problems possible for a restricted

domain. This is often used in ________________.

To find a ____________________ function ___ that ____________________ another function ___, we choose a number ___ in the _______________ of ___ at which ________________. This approximating _________________ is said to be __________________ about ___ or _________________ at ___. The evil plan is to find a polynomial whose ____________ looks like the graph of ____ ____________ this point. If we require that the ____________ of the polynomial function is the ___________ as the slope of the ___________ at ____, then we also have __________________. Using these two requirements we can get a _______________ approximation of ____.

EXAMPLE 1: Consider $f(x)=\frac{x}{x+1}$.

a. Find a first-degree polynomial function $P_1(x)=a_0+a_1x$ whose value and slope agree with the value and slope of f at $x=0$.

x	-0.8	-0.2	-0.1	0	0.1	0.2	1.0
$\frac{x}{x+1}$							
$P_1(x)$							

b. Now find a second-degree polynomial function $P_2(x) = a_0 + a_1x + a_2x^2$ whose value and slope agree with the value and slope of f at $x = 0$.

x	-0.8	-0.2	-0.1	0	0.1	0.2	1.0
$\dfrac{x}{x+1}$	-4	-0.25	-0.1111	0	0.0909	0.16667	0.5
$P_2(x)$							

c. Let's go for a third-degree polynomial function $P_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ whose value and slope agree with the value and slope of f at $x = 0$.

x	-0.8	-0.2	-0.1	0	0.1	0.2	1.0
$\dfrac{x}{x+1}$	-4	-0.25	-0.1111	0	0.0909	0.16667	0.5
$P_3(x)$							

Definition of *n*th Taylor and *n*th Maclaurin Polynomial

If f has n derivatives at c, then the polynomial

is called the ____ ________________ polynomial for ___ at ___.

If ________, then

is also called the ____ __________________ polynomial for ____.

Remainder of a Taylor Polynomial

To __________________ the ___________________ of approximating a function

value _____ by the Taylor polynomial ______, we use the concept of a

____________________ ______.

EXAMPLE 2: Consider the function $f(x) = x^2 \cos x$.

a. Find the second Taylor polynomial for the function $f(x) = x^2 \cos x$ centered at π.

b. Approximate the function at $x = \frac{7\pi}{8}$ using the polynomial found in part a.

Taylor's Theorem

If a function f is differentiable through order $n+1$ in an interval I containing c , then, for each x in I , there exists z between x and c such that

where

A __________________ of this theorem is that

where ________________ is the _________________ value of ____________

between ____ and ___.

For ________ we have

Does this look familiar?

EXAMPLE 3: Use Taylor's Theorem to obtain an upper bound for error of the approximation. Then calculate the exact value of the error.

$$e \approx 1+1+\frac{1^2}{2!}+\frac{1^3}{3!}+\frac{1^4}{4!}+\frac{1^5}{5!}$$

EXAMPLE 4: Determine the degree of the Maclaurin polynomial required for the error in the approximation of the function at the indicated value of *x* to be less than 0.001.

$\cos(0.1)$

9.8: Power Series

When you finish your homework you should be able to…

- π Find the radius and interval of convergence of a power series.
- π Determine the endpoint convergence of a power series.
- π Differentiate and integrate a power series.

WARM-UP: Find the sixth-degree Maclaurin polynomial for $f\left(x\right)=e^{x}$.

This enables us to be able to ________________________ the function

_________ near ________. We found out that the higher the __________ of the

approximating ____________________, the better the approximation becomes.

In this section, you'll see that several important ___________________ can be

represented ____________________ by ________________ series.

Definition of Power Series

If x is a variable, then an infinite series of the form

is called a _____________ series _______________ at _____, where ___ is a constant.

If a power series is ______________ at ____, the power series will be of the form

EXAMPLE 1: Find the power series for $f(x) = e^x$, centered at $x = 0$.

Radius and Interval of Convergence

A power series in ____ can be thought of as a ______________ of ____.

The _______________ of ___ is the _______ of all ___ for which the power series __________________. Every power series converges at its __________.

Therefore, _____ is always in the _______________ of ____. The domain of a power series can take on any one of the following forms:

a ______________ ____________

an ________________

the _______ of ________ numbers

Theorem: Convergence of a Power Series

For a power series centered at c, precisely one of the following is true:

1. The series converges only at ______.
2. There exists a ________ number _______ such that the series converges

   _____________________ for ____________ and diverges for

   ______________.
3. The series converges absolutely for ______ ______.

Endpoint Convergence

Each __________________ must be _____________ for ______________ or

_________________. This results in _____ possible forms an _____________

of _________________ can take on.

_________ 0

_________ _____

_________ ____

Example 2: Find the radius and interval of convergence (including a check for convergence at the endpoints) of the following power series.

a. $\sum_{n=0}^{\infty} (2x)^n$

b. $\sum_{n=0}^{\infty}\frac{\left(3x\right)^{n}}{\left(2n\right)!}$

c. $\displaystyle\sum_{n=0}^{\infty}\frac{(x-3)^{n+1}}{(n+1)4^{n+1}}$

Theorem: Properties of Functions Defined by Power Series

If the function ____

has a radius of convergence of _______, then, on the interval _____________

f is _________________ and thus ______________. The derivative and antiderivative are given below:

1.

2.

The radius of convergence of the series obtained by _________________ or _____________________ a power series is the ___________ as that of the __________________ power series. What may change is the ______________ of convergence.

Example 3: Let $f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$ and $g(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$.

a. Find the interval of convergence of f.

b. Find the interval of convergence of g.

c. Show that $f'(x) = g(x)$.

d. Show that $g'(x) = -f(x)$.

e. Identify the function f.

f. Identify the function g.

Example 4: Write an equivalent series with the index of summation beginning at $n=1$.

a. $\sum_{n=0}^{\infty}(-1)^{n+1}(n+1)x^n$

b. $\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$

9.9: Representing Functions as Power Series

When you finish your homework you should be able to…

- π Manipulate a geometric series to represent a function as a power series
- π Differentiate or integrate a geometric series to represent a function as a power series.

WARM-UP: Find the infinite sum of the convergent series $\sum_{n=0}^{\infty} 5\left(-\frac{3}{4}\right)^{n}$.

Now consider the function $f(x)=\frac{1}{1-x}$.

This ___________________represents $f\left(x\right)=\frac{1}{1-x}$ only on the interval from

________________. What is the domain of f ? _____________________.

How would we represent f on another interval? We must develop a

_________________ _____________ which is ______________ at a different

value.

Example 1: Find the power series for $f\left(x\right)=\frac{1}{1-x}$ centered at $c=-2$.

Example 2: Find a geometric power series for the function $f\left(x\right)=\frac{2}{5-x}$ centered at 0, (a) by manipulating the function into the format of a geometric power series and (b) by using long division.

Example 3: Find a power series for the function, centered at c, and determine the interval of convergence.

a. $f(x)=\dfrac{3}{2x-1},\ c=2$

b. $f\left(x\right)=\dfrac{4}{3x-2},\ c=3$

Operations with Power Series

Let $f(x)=\sum_{n=0}^{\infty} a_n x^n$ and $g(x)=\sum_{n=0}^{\infty} b_n x^n$ be power series centered at 0.

1. $f(kx)=\sum_{n=0}^{\infty} a_n k^n x^n$, where ______ is a __________________.
2. $f(x^N)=\sum_{n=0}^{\infty} a_n x^{nN}$, where ______ is a __________________.
3. $f(x)\pm g(x)=\sum_{n=0}^{\infty}(a_n \pm b_n)$

Note: These operations can change the _________________ of ________________ for the resulting series.

Example 4: Find a power series for the function, centered at *c*, and determine the interval of convergence.

a. $f(x)=\frac{5}{5+x^2}$, $c=0$

b. $f\left(x\right)=\dfrac{3x-8}{3x^{2}+5x-2},\ c=0$

Example 5: Consider the functions $f(x)=\frac{1}{1+x}$ and $g(x)=\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$.

a. Find a power series for f, centered at 0.

b. Use your result from part a to determine a power series, centered at 0, for the function $h(x)=\frac{x}{x^2-1}=\frac{1}{2(1+x)}-\frac{1}{2(1-x)}$. Identify the interval of convergence.

c. Use your result from part a to determine a power series, centered at 0, for the function $r(x) = \frac{2}{(x+1)^3}$. Identify the interval of convergence.

d. Use your result from part a to determine a power series, centered at 0, for the function $s(x) = \ln(1 - x^2)$. Identify the interval of convergence.

9.10: Taylor and Maclaurin Series

When you finish your homework you should be able to…

- π Find a Taylor series or a Maclaurin series for a function.
- π Find a binomial series.
- π Use a basic list of Taylor series to derive other power series.

WARM-UP: Find the 8th degree Maclaurin polynomial for the function $f\left(x\right)=\cos x$.

Now let's see if we can form a power series!

What about that interval of convergence?

Theorem: The Form of a Convergent Power Series

If f is represented by a power series $f(x)=\sum a_n(x-c)^n$ for all x in an open interval I containing c, then

and

Definition of Taylor and Maclaurin Series

If a function f has derivatives of all orders at $x = c$, then the series

is called the _________________ series for _______ at _____. If __________,

then the series is the ____________________ series for ______.

Example 1: Find the Taylor series, centered at c, for the function.

a. $f(x) = e^{-4x},\ c = 0$

b. $f\left(x\right)=\dfrac{1}{1-x},\ c=2$

Theorem: Convergence of Taylor Series

If $\lim_{n\to\infty} R_n = 0$ for all x in the interval I, then the Taylor series for f converges and equals $f(x)$.

Example 2: Prove that the Maclaurin series for $f(x)=\cos x$ converges to $f(x)$ for all x.

Binomial Series

Let's check out the function $f(x)=(1+x)^k$, where k is a rational number. What do you think the Maclaurin series is for this function? Guess what…YOU KNOW HOW TO FIND IT!!! So, on your mark, get set, GO!

1. ________________ $f(x)$ a bunch of times and evaluate each

   ________________ at ______. Evil plan: ___________________ a

   ___________________.

2. Determine the _______________ of _________________...Don't forget to test the _________________!

Guidelines for Finding a Power Series

1. ___________________ $f(x)$ and ________________ each

___________________ at _____ until you find a _____________________.

2. Form the ______________ coefficient __________________, and

determine the _______________ of convergence for the _____________

series.

3. Determine whether the series _________________ to ________ within

the interval of convergence.

Example 3: Find the Maclaurin series for the function using the binomial series.

a. $f(x) = \dfrac{1}{(1+x)^4}$

b. $f\left(x\right)=\sqrt{1+x^3}$

A Basic List of Power Series for Elementary Functions

FUNCTION	INTERVAL OF CONVERGENCE
$\frac{1}{x} =$	$0 < x < 2$
$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots + (-1)^n x^n + \cdots$	$-1 < x < 1$
$\ln x =$	$0 < x \leq 2$
$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^n}{n!} + \cdots$	$-\infty < x < \infty$
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$	$-\infty < x < \infty$
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$	$-\infty < x < \infty$
$\arctan x =$	$-1 \leq x \leq 1$
$\arcsin x =$	$-1 \leq x \leq 1$
$(1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \cdots$	$-1 < x < 1$*

*convergence at endpoints depends on k

Example 4: Find the Maclaurin series for the function using the basic list of power series for elementary functions.

a. $f(x) = \ln(1+x^2)$

b. $f(x) = e^x + e^{-x}$

c. $f\left(x\right)=\cos^2 x$

d. $f\left(x\right) = x\cos x$

Example 5: Find the first four nonzero terms of the Maclaurin series for the function $f(x) = e^x \ln(1+x)$.

Example 6: Use a power series to approximate the value of the integral with an error less than 0.0001.

$$\int_0^{1/2} \arctan x^2 dx$$

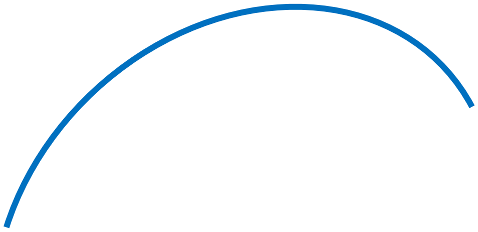
7.4: Arc Length and Surfaces of Revolution

When you finish your homework you should be able to…

π Find the arc length of a smooth curve.
π Find the area of a surface of revolution

Arc length is approximated by ____________ infinitely many ______________.

A _______________ curve is one which has a __________ arc length. A sufficient condition for the graph of a function ____ to be rectifiable between ___________ and _____________ is that _____ be continuous on _____. A function of this type is considered to be ___________________ differentiable on ________ and its graph on the interval _______ is a _________ ________.



Definition of Arc Length

Let the function ____________ represent a smooth curve on the interval ______.

The arc length of ____ between ___ and ___ is

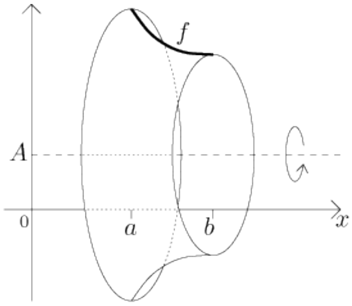
For a smooth curve __________ on the interval ______ the arc length of ____

between ___ and ___ is

EXAMPLE 1: Find the arc length from $(-3,4)$ clockwise to $(4,3)$ along the circle $x^2+y^2=25$. Show that the result is one-fourth the circumference of a circle.

Definition of Surface of Revolution

When the graph of a continuous function is _____________ about a _______,the resulting surface is a ___________ of __________________.



Definition of the Area of a Surface of Revolution

Let the function ____________ have a continuous derivative on the interval ______. The area ____ of the surface of revolution formed by revolving the graph of ___ about a horizontal or vertical axis is

where _____ is the distance between the graph of ___ and the axis of revolution.

If ___________ on the interval _______ then the surface area is

where _____ is the distance between the graph of ___ and the axis of revolution.

EXAMPLE 2: Find the area of the surface generated by revolving the curve $y = 9 - x^2$ about the *y*-axis.

10.1: Conics and Calculus

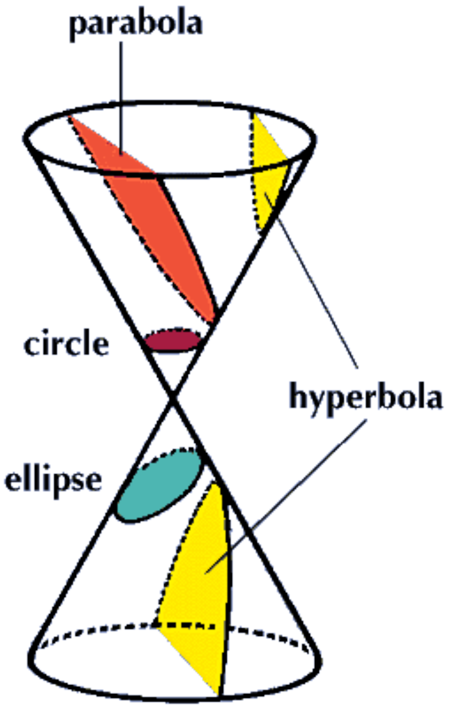
When you finish your homework you should be able to…

- π Use properties of conic sections to analyze and write equations of parabolas, ellipses, and hyperbolas.
- π Classify the graph of an equation of a conic section as a circle, parabola, ellipse, or hyperbola.
- π Find the equations of lines tangent and normal to conic sections

The graph of each type of _____________ section can be described as the

intersection of a plane and two identical ___________ which are connected at

their vertices.



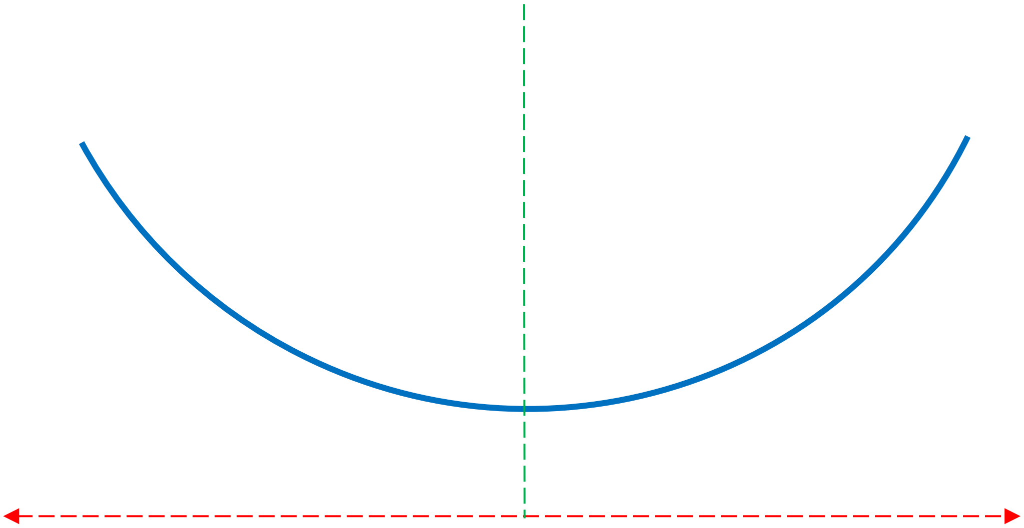
A **parabola** is the set of all

___________________ __________ that are

_______________________ from a fixed line

called the ____________________ and a fixed

point called the ____________________.



Theorem: Standard Equation of a Parabola

The standard form of a parabola with vertex __________ and directrix ________ is

Vertical axis

The standard form of a parabola with vertex __________ and directrix ________ is

Horizontal Axis

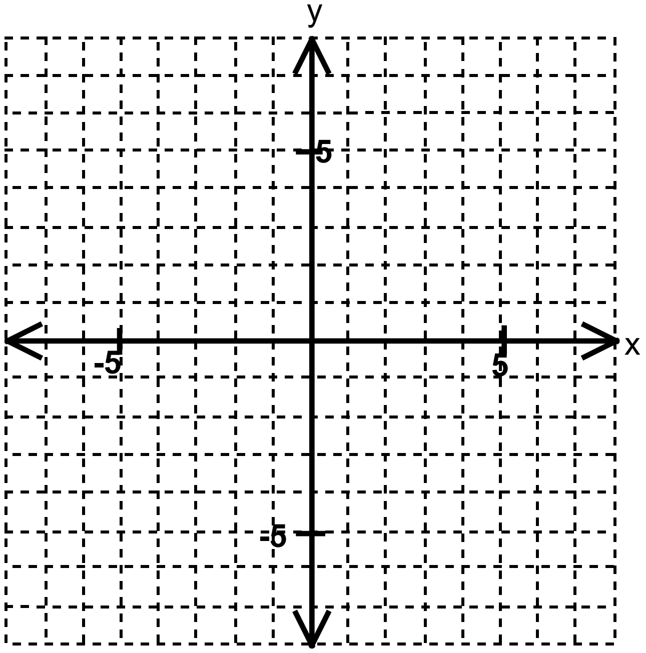
The focus lies on the axis ___ units from the vertex. The coordinates of the focus are

Vertical axis

Horizontal Axis

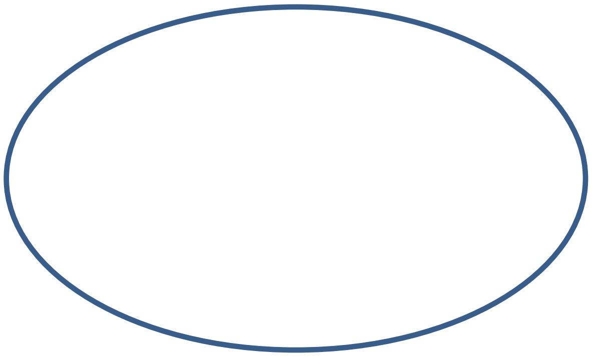
EXAMPLE 1: Consider $y^2 + 6y + 8x + 25 = 0$.

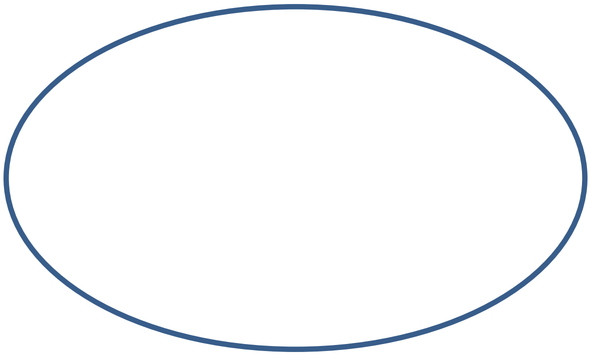
a. Find the vertex, focus, and the directrix of the parabola and sketch its graph.



b. Find the equation of the line tangent to the graph at $x = -4$.

An **ellipse** is the set of all ___________________ __________ the sum of whose distances from two distinct fixed points called _______________ is constant.





Theorem: Standard Equation of an Ellipse

The standard form of the equation of an ellipse with center __________ and major and minor axes of lengths _______ and _______, where ___________, is

Major Axis is Horizontal

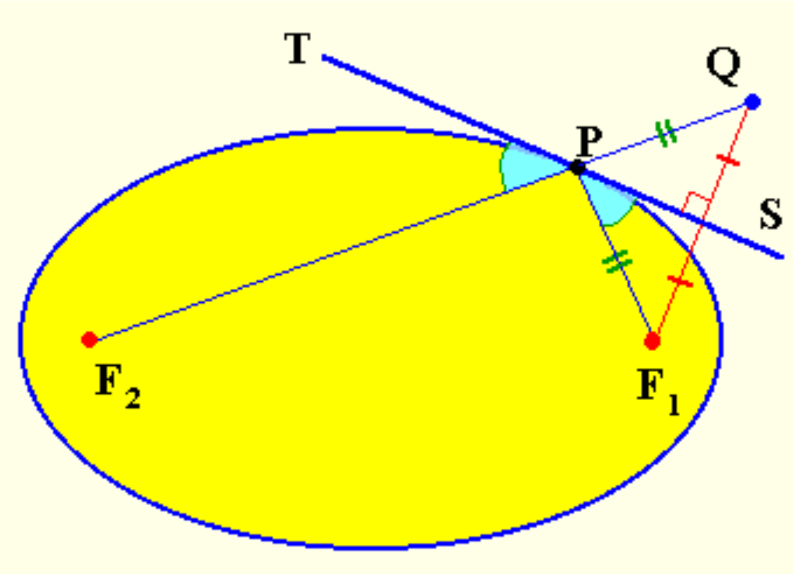
or

Major Axis is Vertical

The foci lie on the major axis, ___ units from the center, with

Theorem: Reflective Property of an Ellipse

Let ____ be a point on an ellipse. The tangent line to the ellipse at point ___ makes __________ angles with the lines through ____ and the _________.



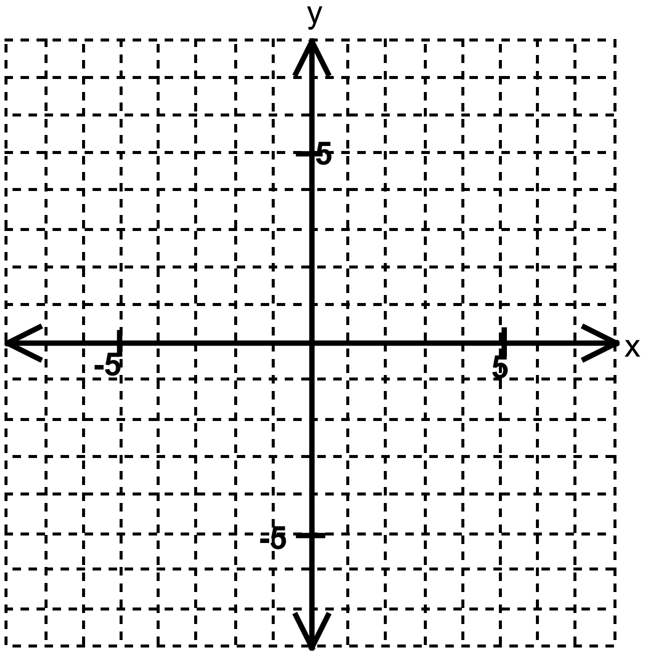
Definition of Eccentricity of an Ellipse

The ____________________ ____ of an ellipse is given by the ratio

For an ellipse that is close to being a _________________, the foci are close to the __________________ and the _______________ ___________ is close to _____________.An ___________________ ellipse has foci which are close to the _________________ and the ____________ ________ is close to ___.

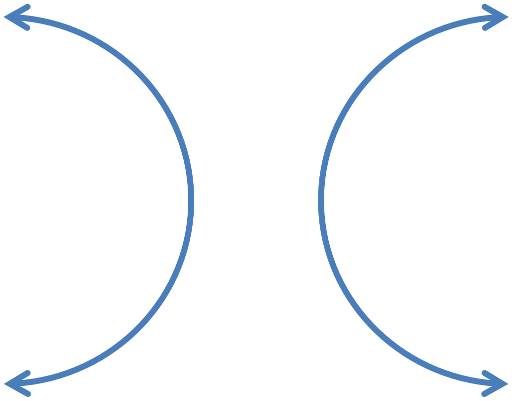
EXAMPLE 2: Consider $16x^2 + 25y^2 - 64x + 150y + 279 = 0$.

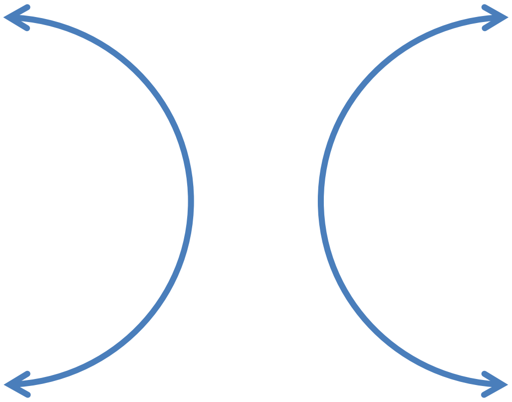
Find the center, foci, vertices, and eccentricity of the ellipse and sketch its graph.



EXAMPLE 3: Find an equation of the ellipse with vertices $(0,3)$ and $(8,3)$ and eccentricity $\frac{3}{4}$.

A **hyperbola** is the set of all ____________________ __________ for which the absolute value of the difference between the distances from two distinct fixed points called ________________ is constant. The line ____________________ connecting the vertices is the __________________ ____________, and the _________________ of the transverse axis is the _______________ of the hyperbola.





Theorem: Standard Equation of a Hyperbola

The standard form of the equation of a hyperbola with center __________ is

Transverse Axis is Horizontal

or

Transverse Axis is Vertical

The vertices are ___ units from the center, and the foci are ____ units from the center with ____________________.

Theorem: Asymptotes of a Hyperbola

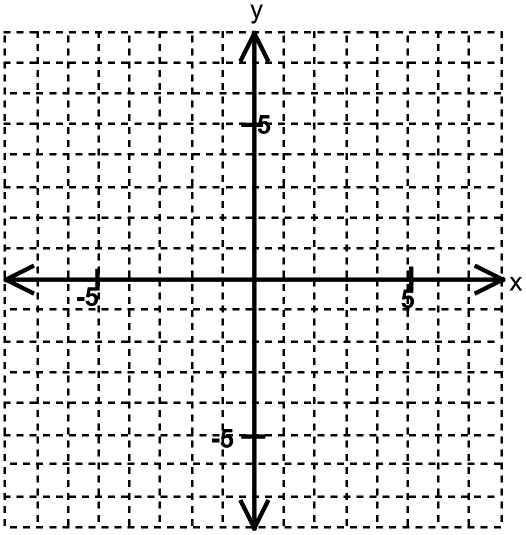
Transverse Axis is Horizontal

or

Transverse Axis is Vertical

EXAMPLE 4: Consider $\frac{y^2}{4} - \frac{x^2}{2} = 1$.

a. Find the center, foci, and vertices of the hyperbola, and sketch its graph using asymptotes.



b. Find equations for the tangent lines to the hyperbola at $x = 4$.

c. Find equations for the normal lines to the hyperbola at $x = 4$.

EXAMPLE 4: A cable of a suspension bridge is suspended in the shape of a parabola between two towers that are 120 meters apart and 20 meters above the roadway. The cable touches the roadway midway between the two towers.

a. Find an equation for the parabolic shape of the cable.

b. Find the length of the cable.

10.2: Plane Curves and Parametric Equations

When you finish your homework you should be able to…

π Sketch the graph of a curve given by a set of parametric equations.
π Eliminate the parameter in a set of parametric equations.
π Find a set of parametric equations to represent a curve.

We currently use a ______________ equation involving _______ variables to represent a _____________. This tells us _____________ an object has been but it doesn't tell us ___________ the object was at a given __________ _______. To determine this _________, we introduce a third variable, ___, called a _________________. Using two equations to represent each ___ and ___ as functions of ___ gives us __________________ _________________.

Definition of a Plane Curve

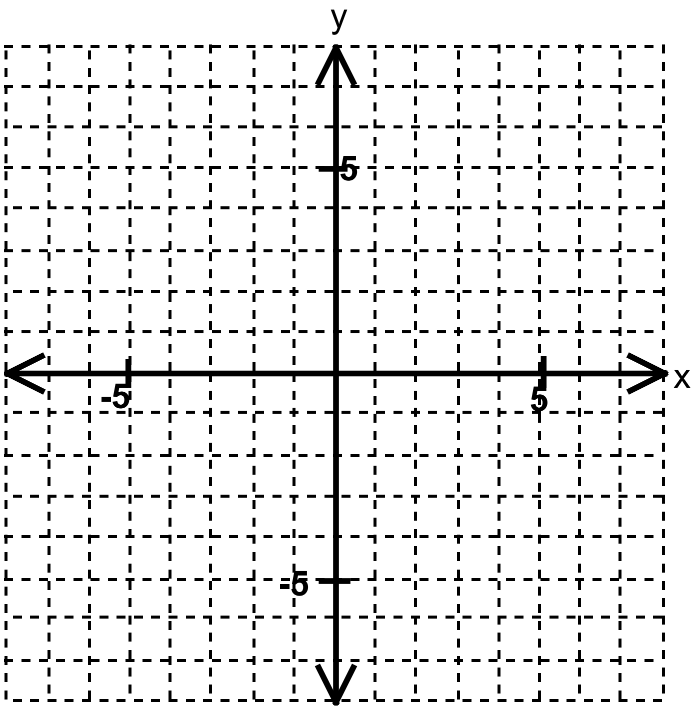
If ___ and ___ are continuous functions of ___ on an interval ___, then the equations

are ________________ equations and ____is the ______________. The set of points _________ obtained as ____ varies over the interval ___ is the ____________ of the parametric equations. Taken together, the _____________ equations and the _________ are a _________ ___________.

EXAMPLE 1: Consider $x = 2t^2$, $y = t^4 + 1$.

a. Sketch the curve represented by the parametric equations. Be sure to indicate the orientation.

t	$x = 2t^2$	$y = t^4 + 1$



b. Write the corresponding rectangular equation by eliminating the parameter.

EXAMPLE 2: Consider $x = \cos\theta$, $y = 2\sin 2\theta$.

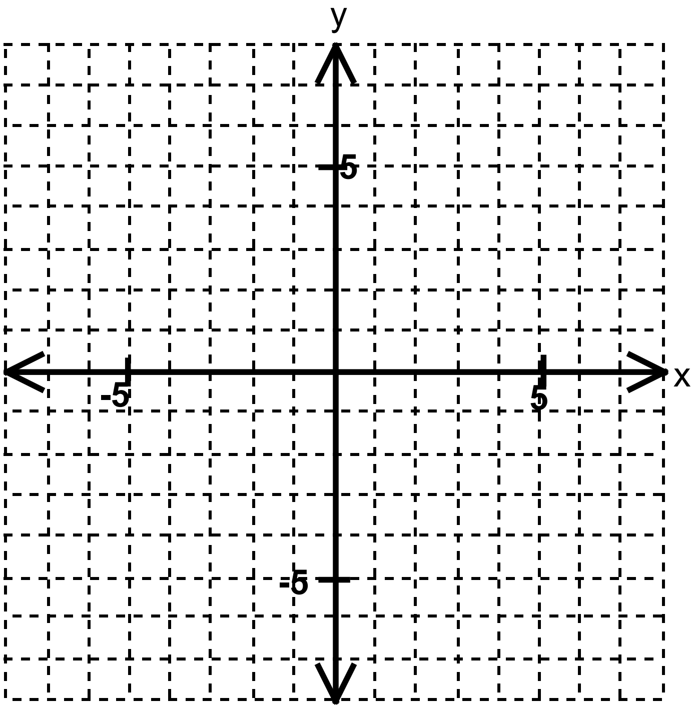
a. Use your graphing calculator to sketch the curve represented by the parametric equations. Be sure to indicate the orientation.

b. Write the corresponding rectangular equation by eliminating the parameter.

EXAMPLE 3: Consider $x = -2 + 3\cos\theta$, $y = -5 + 3\sin\theta$.

a. Sketch the curve represented by the parametric equations. Be sure to indicate the orientation.

θ	$x = -2 + 3\cos\theta$	$y = -5 + 3\sin\theta$



b. Write the corresponding rectangular equation by eliminating the parameter.

EXAMPLE 4: Consider $x = e^{2t}$, $y = e^{t}$.

a. Use your graphing calculator to sketch the curve represented by the parametric equations. Be sure to indicate the orientation.

b. Write the corresponding rectangular equation by eliminating the parameter.

EXAMPLE 5: Find a set of parametric equations for the line or conic.

a. Circle: Center $(-6,2)$, radius 4

b. Ellipse: Vertices $(4,7)$, $(4,-3)$, Foci: $(4,5)$, $(4,-1)$.

10.3: Plane Curves and Parametric Equations

When you finish your homework you should be able to…

- π Find the slope of a line tangent to a plane curve.
- π Find the arc length of a plane curve.
- π Find the area of a surface of revolution given in parametric form.

Theorem: Parametric Form of the Derivative

If a smooth curve *C* is given by the equations

then the slope of *C* at ______ is

EXAMPLE 1: Consider $x = 4\cos t$, $y = 2\sin t$, $0 < t < 2\pi$.

a. Find $\frac{dy}{dx}$.

b. Find $\frac{d^2y}{dx^2}$.

c. Find all points (if any) of horizontal and vertical tangency to the curve.

d. Determine the open t-intervals on which the curve is concave downward or concave upward.

Theorem: Arc Length in Parametric Form

If a smooth curve *C* is given by the equations ___________ and ___________such that *C* does not intersect itself on the interval ______________, except possibly at the endpoints, then the arc length of *C* over the interval is given by

NOTE: Make sure that the arc length is _____________ only once on the interval!!!

EXAMPLE 2: Find the arc length of the curve given by the equations $x = \arcsin t$ and $y = \ln \sqrt{1-t^2}$ on the interval $0 \leq t \leq \frac{1}{2}$.

Theorem: Area of a Surface of Revolution

If a smooth curve *C* is given by the equations ___________ and ___________such that *C* does not intersect itself on the interval ______________, then the area *S* of the surface of revolution formed by revolving *C* about the coordinate axes is given by

Revolution about the *x*-axis; ____________

Revolution about the *y*-axis; ____________

EXAMPLE 3: Find the area of the surface generated by revolving the curve given by the equations $x = 5\cos\theta$ and $y = 5\sin\theta$ on the interval $0 \le \theta \le \pi$ about the *y*-axis.

EXAMPLE 4: A portion of a sphere of radius *r* is removed by cutting out a circular cone with its vertex at the center of the sphere. The vertex of the cone forms an angle of 2θ. Find the surface area removed from the cone.

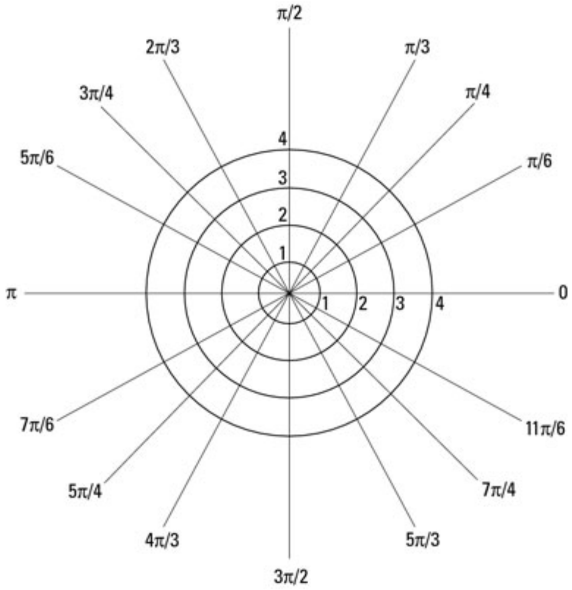
10.4: Polar Coordinates and Graphs

When you finish your homework you should be able to…

π Convert between rectangular and polar coordinates.
π Sketch the graph of an equation in polar form.
π Find the slope of a line tangent to the pole.
π Identify special polar graphs.

Up to this point, we've been using the _______________ coordinate system to sketch graphs. Now we will be using the ___________ coordinate system to sketch graphs given in ____________ form. This form is very useful in the third semester calculus course as it makes many definite _______________ easier to evaluate after switching from rectangular to polar coordinates. The polar coordinate system has a fixed point *O*, called the ________ or ___________. From the pole, an initial ________ is constructe. This is called the _________ axis. Each point *P* in the plane is assigned __________ coordinates in the form _________. ___ represents the ____________ distance from ___ to ___ and ____ is the ____________ angle which is _____________________________ from the polar axis to the segment_____. Unlike rectangular coordinates, each point in polar coordinates does NOT have a ___________ representation. Can you figure out another point in polar coordinates which would be equivalent to

(r,θ)? ________________ How about $(-r,\theta+\pi)$? __________________



In general, the point (r,θ) can be written as

where ___ is any integer. The pole is represented by ______, where ___ is any angle.

Theorem: Coordinate Conversion

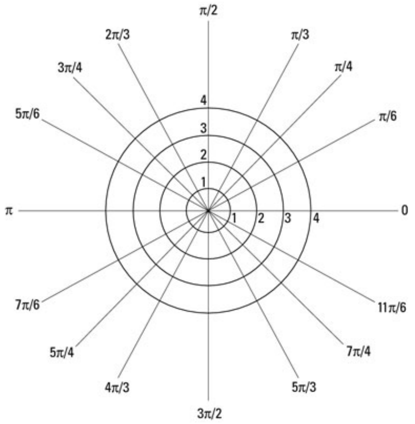
The polar coordinates (r,θ) of a point are related to the rectangular coordinates ___________ of the point as follows:

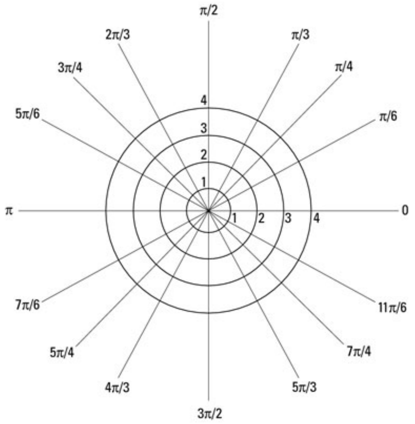
Polar-to-Rectangular **Rectangular-to-Polar**

EXAMPLE 1: Plot the point in polar coordinates and find the corresponding rectangular coordinates for the point.

a. $\left(3, \frac{\pi}{4}\right)$

b. $\left(-2, \frac{5\pi}{3}\right)$





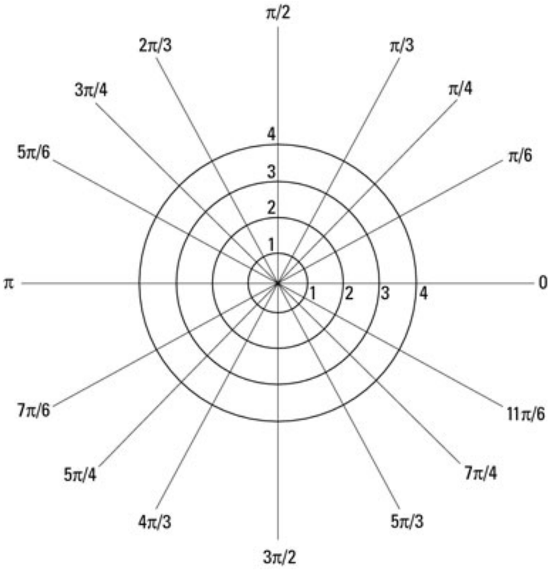
EXAMPLE 2: Find two corresponding polar coordinates for the point given in rectangular coordinates.

a. $\left(3, \frac{\pi}{4}\right)$

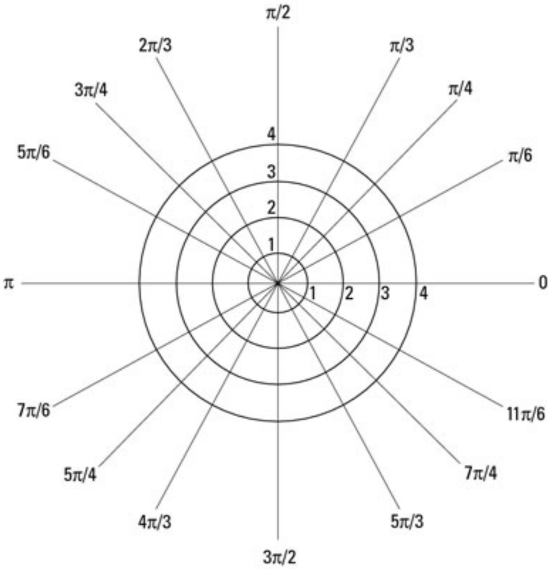
b. $\left(-6, \frac{\pi}{2}\right)$

EXAMPLE 3: Sketch the graph of the polar equation, and convert to rectangular form.

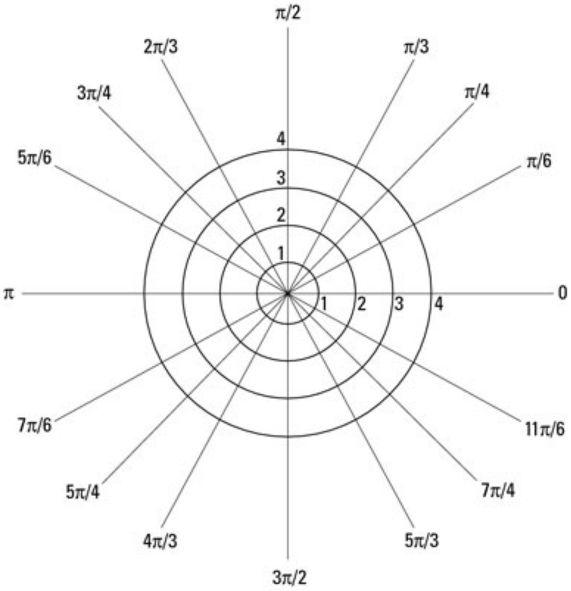
a. $r = -4$



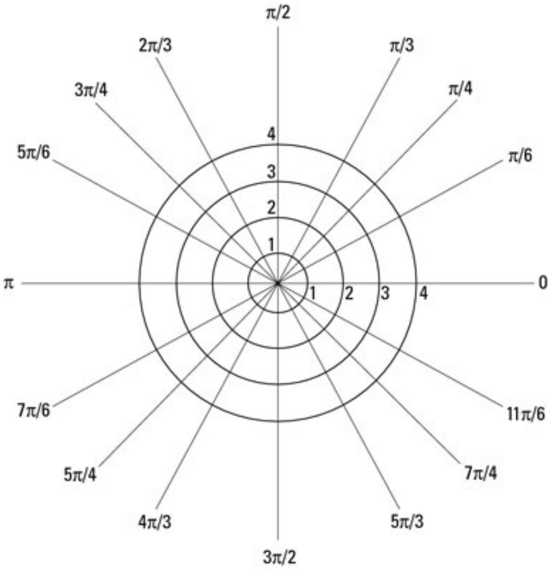
b. $\theta = \dfrac{5\pi}{6}$



c. $r = 3\sin\theta$



d. $r = \cot\theta\csc\theta$



EXAMPLE 4: Convert the rectangular equation to polar form.

a. $x^2 - y^2 = 9$

b. $xy = 4$

Consider $x = r\cos\theta = f(\theta)\cos\theta$ and $y = r\sin\theta = f(\theta)\sin\theta$.

Theorem: Slope in Polar Form

If f is a differentiable function of θ, then the slope of the tangent line to the graph of $r = f(\theta)$ at the point (r,θ) is

provided that ________________ at ___________.

HMMMMM…I guess that means…

Solutions of $\frac{dy}{d\theta} = 0$ **yield** ___________________ **tangents, provided** $\frac{dx}{d\theta} \neq 0$.

Solutions of $\frac{dx}{d\theta} = 0$ **yield** ___________________ **tangents, provided** $\frac{dy}{d\theta} \neq 0$.

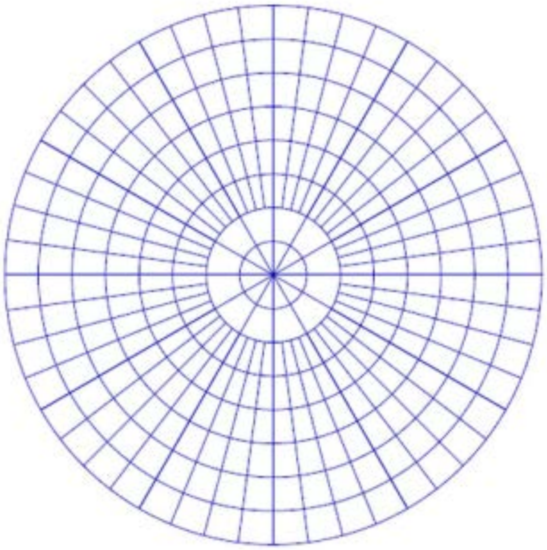
If $\frac{dy}{d\theta} = 0$ **and** $\frac{dx}{d\theta} = 0$ **simultaneously, then no conclusion can be drawn about** ____________ **lines.**

Theorem: Tangent Lines at the Pole

If $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, then the line $f(\alpha) = 0$ is _______________ at the ________ to the graph of ____________.

EXAMPLE 5: Consider $r = 2(1 - \sin\theta)$. Hint: use $\frac{\pi}{24}$ for the increment between the values of θ.

a. Sketch the graph of the equation.



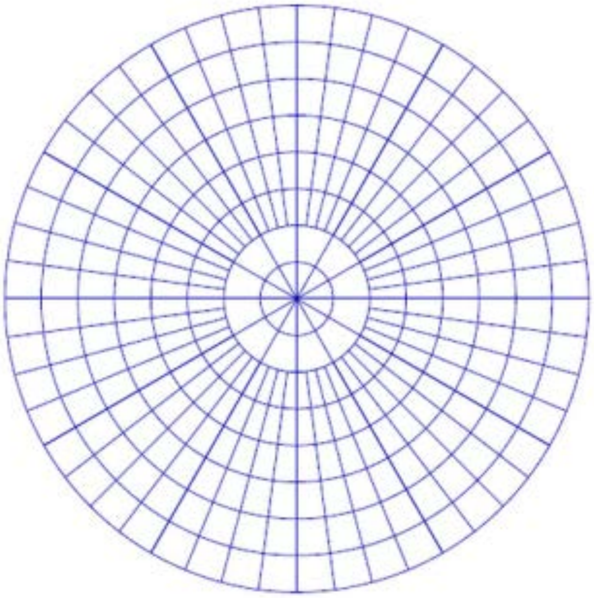
b. Find $\frac{dy}{dx}$.

c. Find all points (if any) of horizontal and vertical tangency to the curve.

d. Find the tangents at the pole.

EXAMPLE 6: Consider $f(\theta) = 8\cos 3\theta$.

a. Graph the equation by hand.



b. Find $\frac{dy}{dx}$.

c. Find all points (if any) of horizontal and vertical tangency to the curve.

d. Find the tangents at the pole.

10.5: Area and Arc Length in Polar Coordinates

When you finish your homework you should be able to…

- π Find the points of intersection between polar graphs.
- π Find the area of a region bounded by a polar graph.
- π Find the arc length of a polar graph.
- π Find the area of a surface of revolution (polar form)

To find the points of ____________________ of polar graphs, you merely

__________ the ____________ of ________________ equations.

EXAMPLE 1: Find the points of intersection of the graphs of the equations $r = 3(1+\sin\theta)$ and $r = 3(1-\sin\theta)$.

The formula for the __________ of a __________ region is developed by

______________ infinitely many _________ of _________. Recall that the

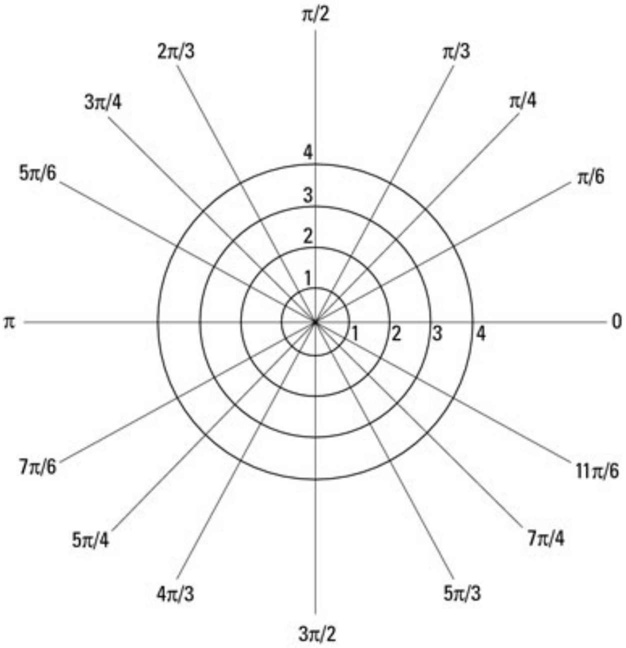
area of a sector is _______________.

Theorem: Area in Polar Coordinates

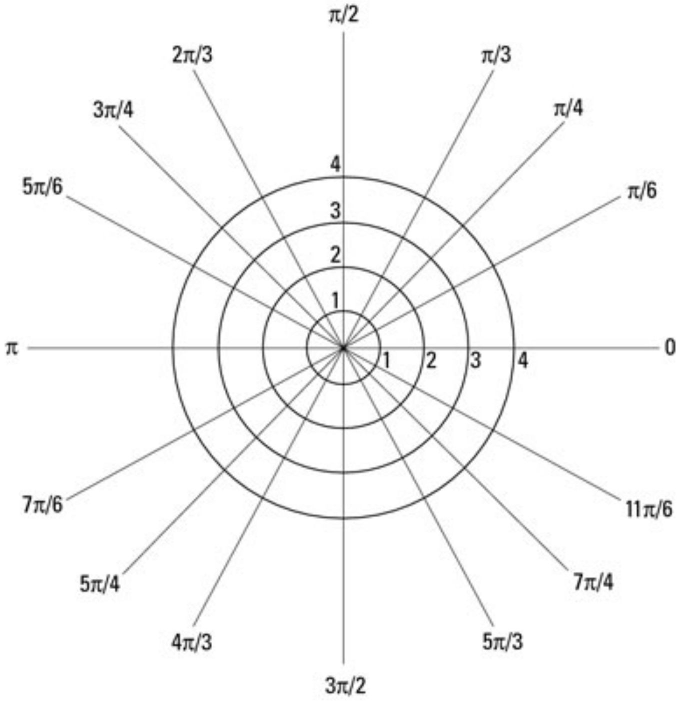
If f is continuous and nonnegative on the interval $[\alpha, \beta]$, $0 < \beta - \alpha \leq 2\pi$, then the area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is

$0 < \beta - \alpha \leq 2\pi$

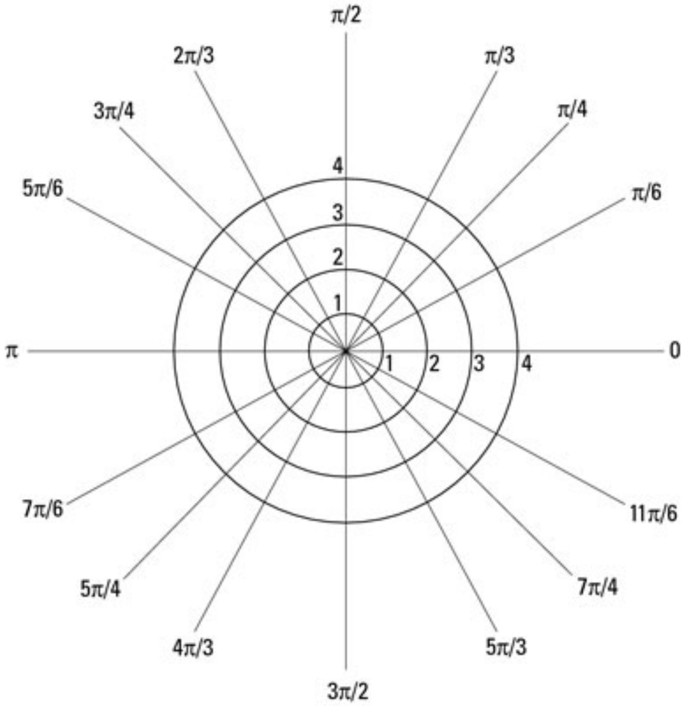
EXAMPLE 2: Find the area of the region of one petal of $r = 4\sin 3\theta$.



EXAMPLE 3: Find the area of the region of the interior of $r = 4 - 4\cos\theta$.



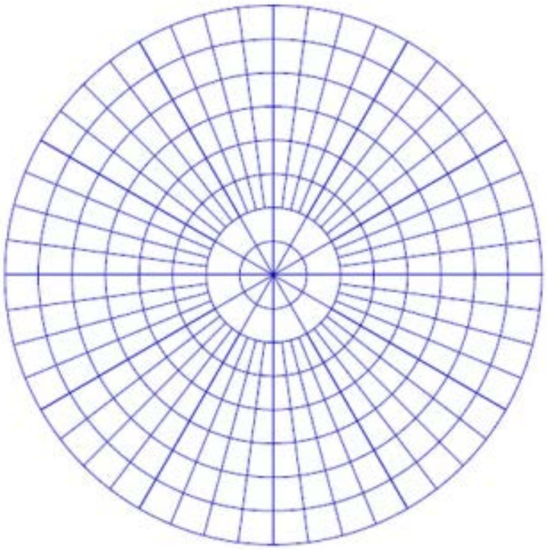
EXAMPLE 4: Find the area of the common interior of $r = 2(1 + \cos\theta)$ and $r = 2(1 - \cos\theta)$.



Theorem: Arc Length of a Polar Curve

Let f be a function whose derivative is continuous on an interval $\alpha \le \theta \le \beta$. The length of the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

EXAMPLE 5: Find the arc length of the curve $r = 8(1+\cos\theta)$ over the interval $0 \le \theta \le 2\pi$.



Theorem: Area of a Surface of Revolution

Let f be a function whose derivative is continuous on an interval $\alpha \le \theta \le \beta$. The area of the surface formed by revolving the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ about

the polar axis is:

the line $\theta = \frac{\pi}{2}$ is:

EXAMPLE 6: Find the area of the surface formed by revolving the curve $r = 6\cos\theta$ about the polar axis over the interval $0 \le \theta \le \frac{\pi}{2}$.